Dynamice of Three Unidirectionally Coupled Duffing Oscillators

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This article presents a system of three unidirectionally coupled Duffing oscillators. On the basic of numerical study we show the mechanism of translation from steady state to chaotic and hyperchaotic behavior. Additionally, the impact of parameter changes on the behavior of the system will be presented. Confirmation of our results are bifurcation diagrams, timelines, phase portraits, Poincare maps and largest Lyapunov exponent diagrams.

Keywords: Oscillator, bifurcation diagram, phase portrait, Poincare map

1. Introduction

Recently, we observed a growing interest in the theory of nonlinear dynamic systems. Rapid development of this branch of science has caused penetration of nonlinear dynamics to other areas research, such as biology, economics, chemistry, mechanics Quantum. Analysis of the dynamics of coupled nonlinear oscillators is one area that has not yet been thoroughly investigated. Therefore, this article aims presentations to the conservation of these oscillators – analysis emerging types of attractors in phase space following a change in coupling coefficient.

2. Model

The subject of our research is a system of three unidirectionally coupled Duffing oscillators [1] shown in Fig. 1. Each of the three oscillators is described by the following equation:

$$\ddot{z} + C_1\dot{z} + C_2 z + C_3 z^3 = 0 \tag{1}$$

where C_1, C_2, C_3 are constant values. These constant values are constant in all numerical studies: $C_1 = 0,1$, $C_2 = -0,1$, $C_3 = 1$.
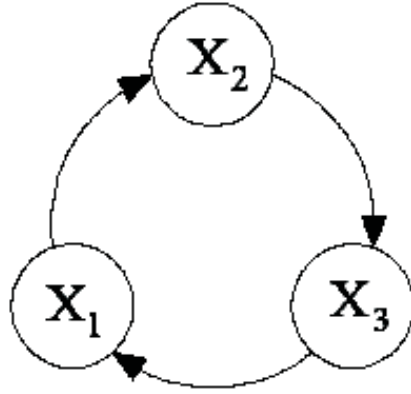
Figure 1 System of three unidirectionally coupled Duffing oscillators

By introducing new coordinates $x = z$, $y = \dot{z}$ to equation (1) a system of three oscillators shown in Fig. 1 can be described by the equations [2]:

$$
\begin{aligned}
\dot{x_1} &= y_1 \\
\dot{y_1} &= -C_1 y_1 - C_2 x_1 - C_3 x_1^3 + C_4 (x_3 - x_1) \\
\dot{x_2} &= y_2 \\
\dot{y_2} &= -C_1 y_2 - C_2 x_2 - C_3 x_2^3 + C_4 (x_1 - x_2) \\
\dot{x_3} &= y_3 \\
\dot{y_3} &= -C_1 y_3 - C_2 x_3 - C_3 x_3^3 + C_4 (x_2 - x_3)
\end{aligned}
\tag{2}
$$

where C_4 is the coupling coefficient, whose value change from 0 to 1.

All numerical studies were carried out for the same initial conditions:

$$
\begin{aligned}
x_1 &= 1, & y_1 &= 3 \\
x_2 &= 5, & y_2 &= 2 \\
x_1 &= 3, & y_3 &= 4
\end{aligned}
$$

3. Numerical analysis of three identical, unidirectionally coupled Duffing oscillators

At this point we present the mechanism that leads to a translation from steady state to a non–periodical oscillations of three identical, unidirectionally coupled Duffing oscillators. Using equations (2), we analyze the behavior of changing the coupling coefficient C_4.

Bifurcation diagram (Fig. 2) and largest Lyapunov exponent diagrams (Fig. 3) are as follows [3], [4].

For the coupling coefficient $C_4 \leq 0,0057$ we see the pursuit of stationary point (Fig. 4a), while the value of $C_4 = 0,0057$ we can observe the phenomenon of

transitional chaos (Fig. 4b). All the values of Lyapunov exponents are negative ($\lambda < 0$). With the increase of the coupling coefficient C4 from 0,0058 to 0,2 as a result of subcritical Hopf bifurcation, we observe chaotic behavior (Fig. 5a, 6a, 7a). The largest Lyapunov exponent is positive ($\lambda_{max} > 0$). Further increase of the parameter causes the transition to hyperchaotic motion (Fig. 5b, 6b, 7b), which is characterized by the Lyapunov exponents that have two positive values.
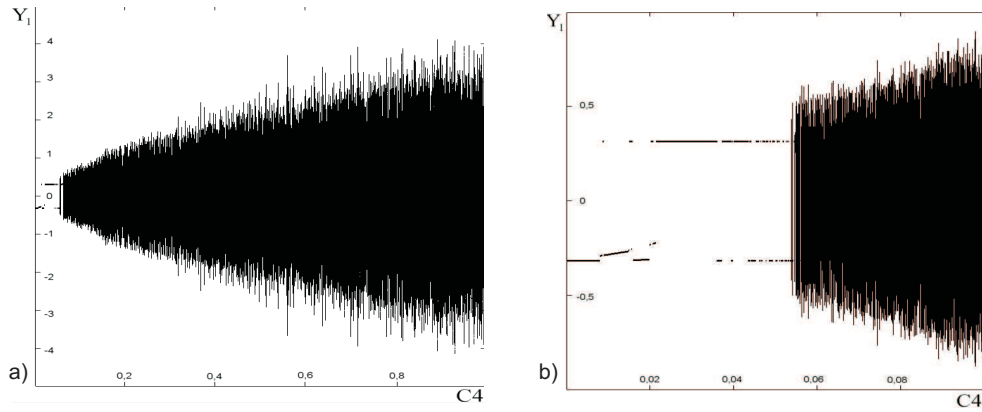

Figure 2 a) Bifurcation diagram for $C_4 = 0 \div 1$, b) Bifurcation diagram for $C_4 = 0 \div 0,1$
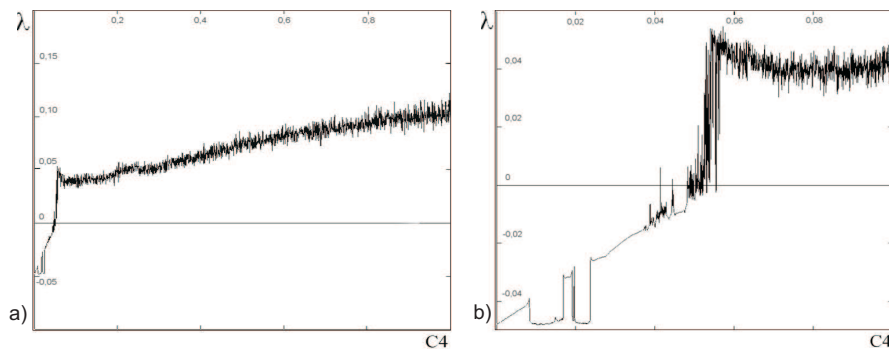

Figure 3 a) The largest Lyapunov exponent for $C_4 = 0 \div 1$, b) The largest Lyapunov exponent for $C_4 = 0 \div 0,1$
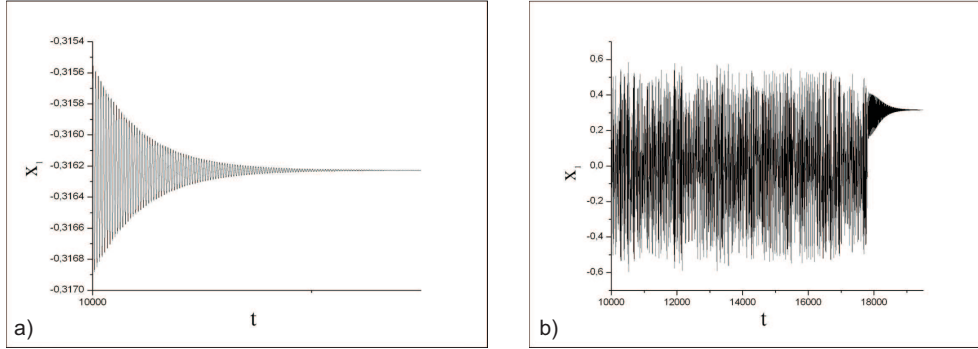
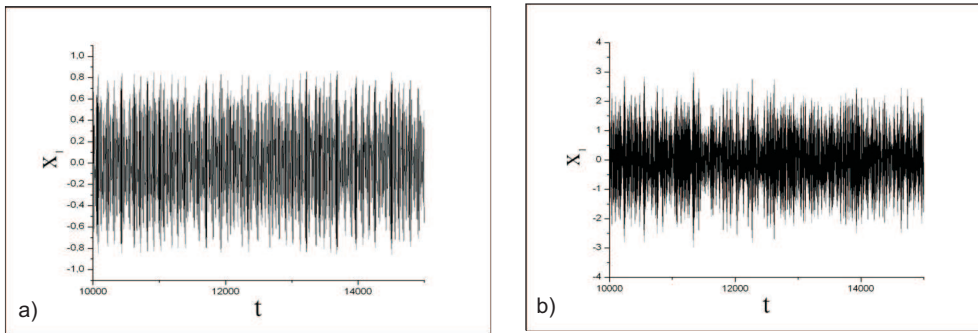**Figure 4** a) Timelines for $C_4 = 0,0056$, b) Timelines for $C_4 = 0,0057$

Figure 5 a) Timelines for $C_4 = 0,1$, b) Timelines for $C_4 = 0,5$
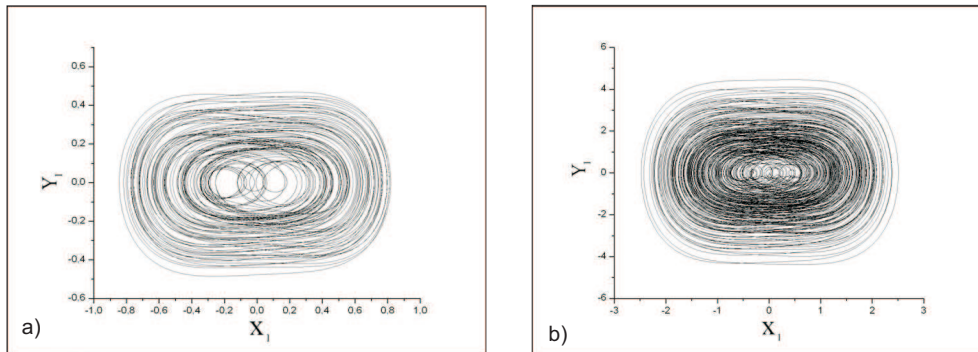

Figure 6 a) Phase portrait for $C_4 = 0,1$, b) Phase portrait for $C_4 = 0,5$
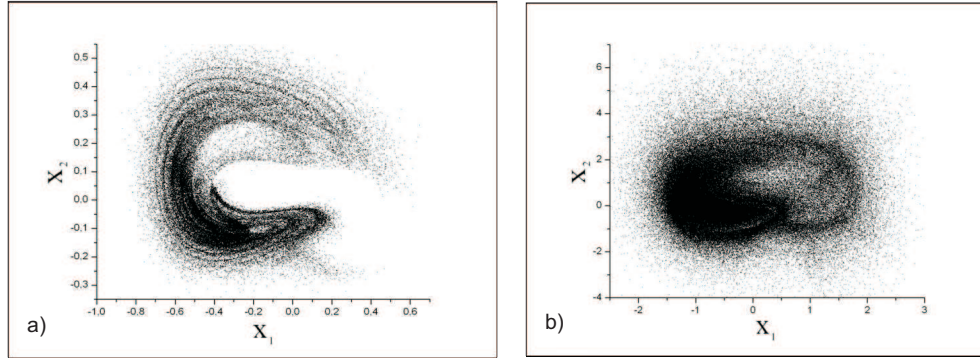
Figure 7 a) Pioncare map for $C_4 = 0, 1$, b) Pioncare map for $C_4 = 0, 5$

4. Numerical analysis of three non–identical, unidirectionally coupled Duffing oscillators

In the case of non–identical, unidirectionally coupled Duffing oscillators equations take the form:

$$
\begin{aligned}
\dot{x_1} &= y_1 \\
\dot{y_1} &= -C_1 y_1 - C_2 x_1 - C_3 x_1^3 + C_4(x_3 - x_1) \\
\dot{x_2} &= y_2 \\
\dot{y_2} &= -(C_1 + C_5)y_2 - C_2 x_2 - C_3 x_2^3 + C_4(x_1 - x_2) \\
\dot{x_3} &= y_3 \\
\dot{y_3} &= -(C_1 + C_6)y_3 - C_2 x_3 - C_3 x_3^3 + C_4(x_2 - x_3)
\end{aligned} \tag{3}
$$

where C_5 and C_6 are additional constants, introducing variations in the system. Are, respectively: $C_5 = 0, 001$, $C_6 = 0, 002$.

Bifurcation diagram and largest Lyapunov exponent diagrams are shown in Figs 8 and 9.

For the coupling coefficient $C_4 < 0, 0057$ (Fig. 10), we can see the pursuit of the stationary point, all Lyapunov exponents are negative ($\lambda < 0$). When the coupling coefficient $C_4 \geq 0, 0057$ is subcritical Hopf bifurcation, we observe the chaotic behavior of the system investigated (Figs 11a, 12a,13a). The largest Lyapunov exponent has a positive value ($\lambda_{max} > 0$). Increasing C_4 above 0,2 is moving towards the system in hyperchaos (Fig. 11b, 12b, 13b) – two Lyapunov exponents are larger than zero.
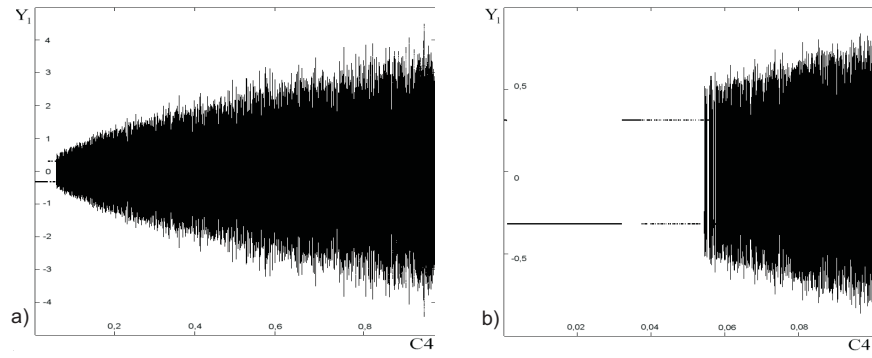
Figure 8 a) Bifurcation diagram for $C_4 = 0 \div 1$, b) Bifurcation diagram for $C_4 = 0 \div 0,1$
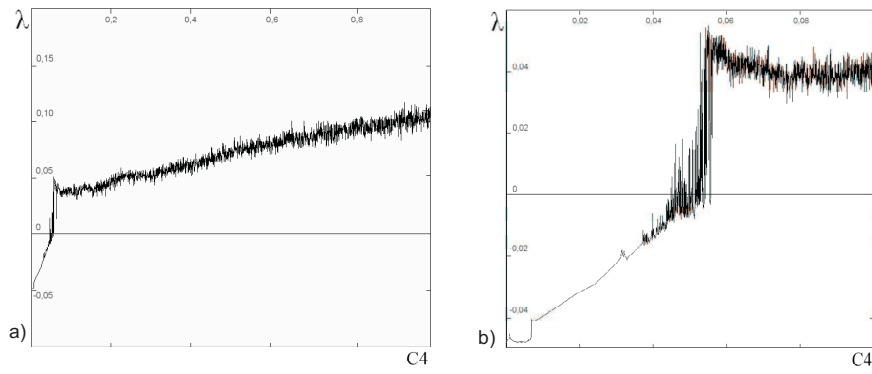

Figure 9 a) The largest Lyapunov exponent for $C4 = 0 \div 1$, b) The largest Lyapunov exponent for $C_4 = 0 \div 0,1$
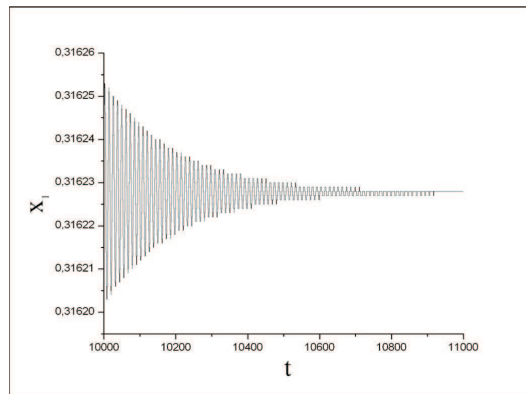
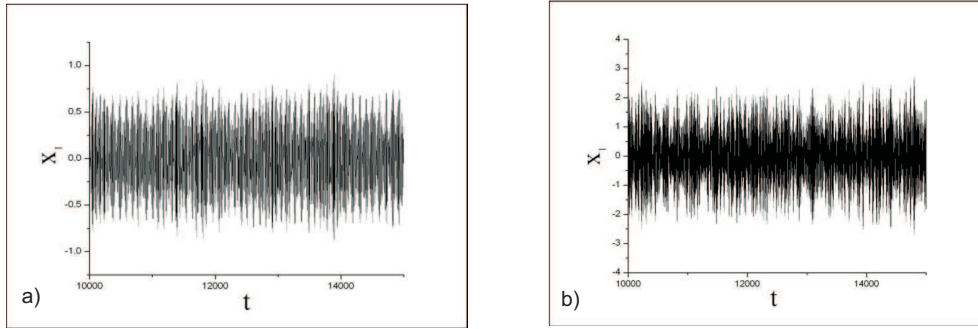

Figure 10 Timelines for $C_4 = 0,0056$

Figure 11 a) Timelines for $C_4 = 0, 1$, b) Timelines for $C_4 = 0, 5$
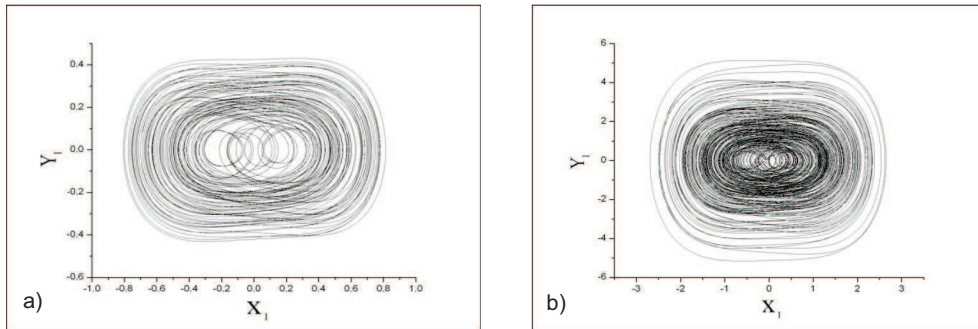

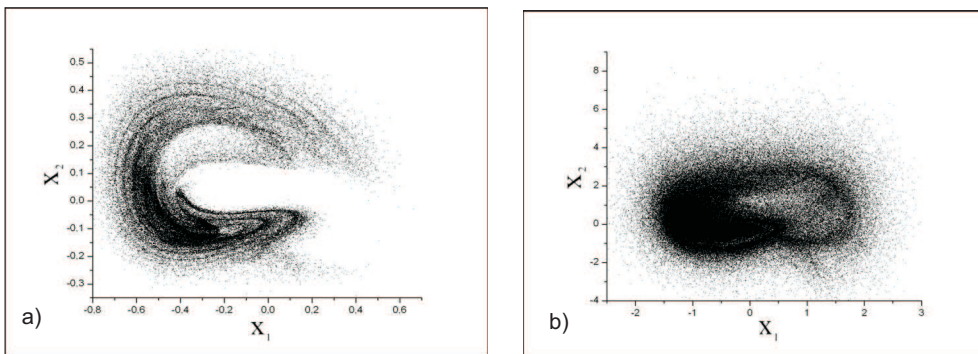**Figure 12** a) Phase portrait for $C_4 = 0, 1$, b) Phase portrait for $C_4 = 0, 5$

Figure 13 a) Pioncare map for $C_4 = 0, 1$, b) Pioncare map for $C_4 = 0, 5$
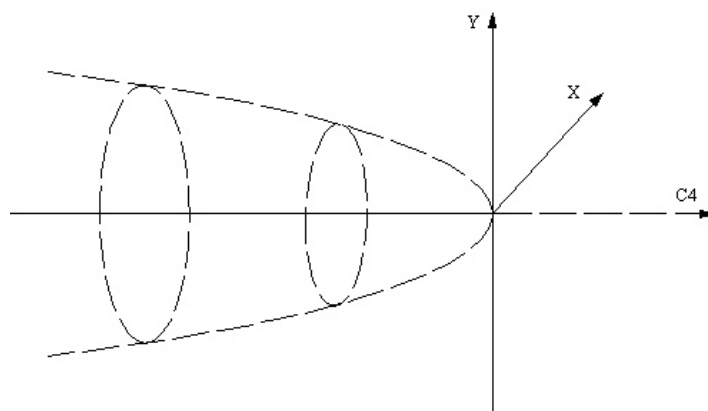
Figure 14 Subcritical Hopf bifurcation

5.   Conclusions

Research show that the introduction of a small parameter mismatch to coupled oscillators has little influence of the results. Decisive influence on the obtained values is coupling coefficient C4.

This is critical from the standpoint of experimental studies, where you have to take into account some differences in values of conjugated system – nominal terms, they should be identical.

For small values of coupling coefficient, the system tends to the critical point. By increasing the coupling coefficient we observe chaotic motion, the value of 0,2 (two positive Lyapunov exponents) starts hyperchaos.

The analysis show that the transition from steady state in the chaotic motion is a result of subcritical Hopf bifurcation (Fig. 14). – a stable fixed point and unstable orbits at the bifurcation changed from an unstable stationary point.

References

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