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Determination of the Classical Orbital Elements of the Perturbed and Perturbing Planets from First Principles. Part I: Outline

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In this outline we present a rather simple method to solve the planetary perturbation problem. We do not avoid the introduction of the expansion of the planetary disturbing function, the formulae of the elliptic expansions and the truncation of the Poisson series at the desired degree. We should remark that all orders of magnitude of the masses of the planets are taken into consideration, which is a very important result of this approach – which we encounter in the order by order approach of planetary theory.

Keywords: Celestial mechanics, orbital mechanics, planetary theory

1. Equation of Motion

It is well known that the equation of motion of the perturbed planet is given by:

$$\ddot{x} + \mu x/r^3 = \partial R/\partial x \tag{1}$$

Where

$\mu = G\,(m_0 + m) = k^2\,(m_0 + m)$,

m_0 – mass of the Sun,

m – mass of the disturbed planet,

R – perturbing function associated with the perturbing planet P'.

We may write for the action of m' on m

$$R = k^2 m' \left\{\Delta^{-1} - (xx' + yy' + zz') \times r'^{-3}\right\} \tag{2}$$

and

$$R' = k^2 m \left\{\Delta^{-1} - (xx' + yy' + zz') \times r^{-3}\right\} \tag{3}$$

for the action of m on m' ; $r' > r$.

From Eq. (1), we may write

$$\ddot{x} + \mu x r^{-3} = \frac{\partial}{\partial x}\left[k^2 m' \left\{\Delta^{-1} - (xx' + yy' + zz')/r'^3\right\}\right]$$
$$\ddot{x} = -\mu x r^{-3} + k^2 m' \frac{\partial}{\partial x}\left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{-1/2}$$
$$-k^2 m' \frac{\partial}{\partial x}\left[(xx' + yy' + zz')\left(x'^2 + y'^2 + z'^2\right)^{-3/2}\right]$$

i.e.

$$\begin{aligned} \ddot{x} &= -\mu x r^{-3} + k^2 m' \Delta^{-3}(x' - x) - k^2 m' x' r'^{-3} \\ \ddot{y} &= -\mu y r^{-3} + k^2 m' \Delta^{-3}(y' - y) - k^2 m' y' r'^{-3} \\ \ddot{z} &= -\mu z r^{-3} + k^2 m' \Delta^{-3}(z' - z) - k^2 m' z' r'^{-3} \end{aligned} \tag{4}$$

Eqs (4) describe the revolution of the 2 planets (disturbed and disturbing) in elliptic orbits with the primary (the Sun) in focus (1), (2).

2. Elliptic Expansions:

We have

$$\left(\frac{x}{r}\right)\left(\frac{r}{a}\right) = \frac{x}{a} i.e. x = a\left(\frac{x}{r}\right)\left(\frac{r}{a}\right) \tag{5}$$

$$\begin{aligned} \frac{x}{r} &\approx \cos(\varpi + M) + e\left[\cos(\varpi + 2M) - \cos\varpi\right] \\ &+e^2\left[\frac{9}{8}\cos(\varpi + 3M) - \frac{1}{8}\cos(\varpi - M) - \cos(\varpi + M)\right] \\ &+\sin^2\frac{I}{2}\left[\cos(\varpi - 2\Omega + M) - \cos(\varpi + M)\right] \end{aligned} \tag{6}$$

$$\begin{aligned} \frac{x}{r} &\approx \sin(\varpi + M) + e\left[\sin(\varpi + 2M) - \sin\varpi\right] \\ &+e^2\left[\frac{9}{8}\sin(\varpi + 3M) - \frac{1}{8}\sin(\varpi - M) - \sin(\varpi + M)\right] \\ &-\sin^2\frac{I}{2}\left[\sin(\varpi - 2\Omega + M) + \sin(\varpi + M)\right] \end{aligned} \tag{7}$$

$$\frac{z}{r} \approx 2\sin\frac{I}{2}\sin(\varpi - \Omega + M) + 2e\sin\frac{I}{2}\left[\begin{array}{c} \sin(\varpi - \Omega + 2M) \\ -\sin(\varpi - \Omega) \end{array}\right] \tag{8}$$

Where $\varpi = \omega + \Omega$ and I one of the orbital element which represents the inclination of the orbital plane upon the ecliptic.

In the formulas (6 – 8), we truncate at the third order of magnitude of e, $\sin\frac{I}{2}$.

Similarly

$$y = a\left(\frac{y}{r}\right)\left(\frac{r}{a}\right) \qquad z = a\left(\frac{z}{r}\right)\left(\frac{r}{a}\right) \tag{9}$$

The equality for $\frac{r}{a}$ is given by

$$\frac{r}{a} \approx 1 - e\cos M + \frac{1}{2}e^2(1 - \cos 2M) \tag{10}$$

Evidently from formula (10), we are capable of finding easily the expression for r^{-3}, r'^{-3} by the application of the binomial theorem.(3)

3. Expansion of Disturbing Function

We can derive Δ^{-3} from the expression Δ^{-s} by putting $s = 3$, using the method of differential operators, whence

$$\Delta^{-3} = a'^{-3} \rho^D \rho'^{-D-3} \left(1 + \alpha^2 - 2\alpha \cos\theta\right)^{-3/2} \tag{11}$$

$$\Delta^2 = r^2 + r'^2 - 2rr' \cos\theta \tag{12}$$

θ is the angle between vector radii r, r' and

$$\begin{aligned} \alpha &= \frac{a}{a'} \qquad D = \alpha \frac{d}{d\alpha} \\ \rho &= \frac{\left(1 - e^2\right)}{(1 + e\cos f)} \\ \rho' &= \frac{\left(1 - e'^2\right)}{(1 + e'\cos f')} \end{aligned} \tag{13}$$

f, f' are the true anomalies of the two elliptic orbits.

The above equations (11 – 13) enable us to find the expression for Δ^{-3} in terms of the classical orbital elements, using the elliptic motion expansions in terms of the mean anomaly M instead of the true anomaly f (4, 5, 6).

Whence we can evaluate $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ Eq. (4), from the above series of formulas.

4. Discussion and future work

In part II, we shall extend the expansion up to higher degree in eccentricity and inclination to get more precise evaluations. More details especially for the derivation of Δ^{-s} will be cited. Moreover integration with respect to physical time of $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ will be implemented to find the components of the velocities $\dot{x}$, $\dot{y}$, $\dot{z}$ and once more to get the coordinates itself x, y, z. From the perturbed x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$, we are capable of the assignment the perturbed classical orbital elements of the two planets a, e, i, ϖ, Ω, M at any epoch. We take into account all orders of magnitudes of the masses of the 2 planets, which is a very important result, reducing the construction of the solution of the problem to the half. Thus we have closed formulae for the masses, but open ones for the eccentricity – inclination.

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