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Optimal Design of Shells of Uniform Stability Stiffened by Ribs

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The problem of optimal design of rotationally symmetrical shells of uniform stability stiffened by ribs was discussed in the current paper. The structure was modelled in ANSYS software and solved by FEM method. The formulation of uniform local stability was successfully verified by the linear buckling solution. The optimization tasks were solved numerically using the modified Particle Swarm Optimization algorithm. The critical loading multiplier was increased by determining the optimal shape of the meridian, distribution of a wall thickness in a coat of shell and the placement of ribs inside the shell.

Keywords: Optimal design, stability of shells, Particle Swarm Optimization, finite elements.

1. Introduction

The optimization of shells against instability is rather complex task. The papers devoted to this problem have received considerable attention for several decades. The problem of shell with a double non-negative curvature is taken into account in this work. Moreover, the concept of a shell of uniform stability was applied. The shell of uniform stability is one in which "the condition of local stability is satisfied in the form of equality not only at the dangerous point but at any point of the shell" – quoted from [1]. Due to the proposed linear theory of stability, neither nonlinear behaviors, nor geometrical imperfections of the structure, were taken into account. In the optimization problem of such a shell the value of loading multiplier is the objective function, whereas constraints are the volume of material and the capacity of a shell. So formulated optimization task of shells – however not stiffened – were solved by e.g. [2, 3, 4].

The Particle Swarm Optimization (PSO) is one of the most popular stochastic search method. This evolutionary computation technique was introduced by [5]. Since that time the increasing interest in this method can be observed. The modified PSO algorithm (MPSO) was successively applied by [6] for solving engineering tasks e.g. optimal design of columns against instability constraints and simple structures against postcritical behavior. The MPSO is adapted to cope with constrained nonlinear optimization tasks for discrete and continuous design variables. The examples of design of structural elements can be found in the paper by [7].

As a tool to solve analysis task the ANSYS software was used. The model created in script language gives a possibility to easy extend the task for stiffened or composite shells. Such problems are very difficult to solve by analytical methods. The MPSO algorithm was coded in C++ language and built to separate executable file. This application calls an analysis file coded in Ansys Parametric Design Language (APDL) on the subsequent steps of iteration.

2. Shell of uniform stability

The considered rotationally symmetrical shell with doubly non–negative curvatures and cylindrical reference shell in background are presented in Fig. 1. The geometry of a reference shell is described by the following symbols: R_0 stands for the radius of a shell, L_0 means the half of its length and H_0 thickness of its wall. These values are constant. The geometry of a shell which is looked for in the optimization task is described by the following symbols: R and H are the radius and wall thickness, R_ϑ and R_φ are the radii of the circumferential and meridional curvatures, respectively. The shell is loaded by an external hydrostatic pressure p.

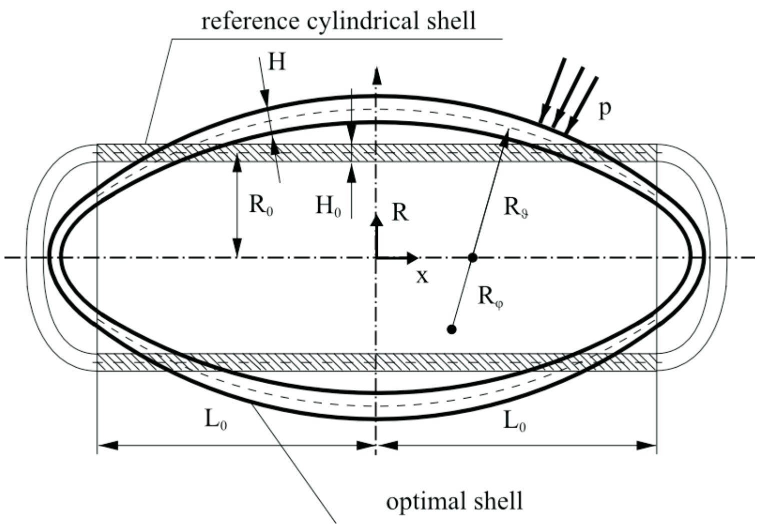

Figure 1 Rotationally symmetrical shell subjected to external pressure

The meridional and circumferential radii are given by the formulas:

$$R_\varphi = \frac{\left(1+R'^2\right)^{3/2}}{-R''} \tag{1}$$

$$R_\vartheta = R\left(1+R'^2\right)^{1/2} \tag{2}$$

For further calculations it is more convenient to introduce the following dimensionless quantities:

$$\begin{array}{llll} r = \dfrac{R}{R_0} & r_\vartheta = \dfrac{R_\vartheta}{R_0} & r_\varphi = \dfrac{R_\varphi}{R_0} & k = \dfrac{R_\varphi}{R_\vartheta} \\ \xi = \dfrac{x}{L_0} & \zeta = \xi \cdot 100 & h = \dfrac{H}{H_0} & \mu = \dfrac{R_0}{L_0} \end{array} \tag{3}$$

It is assumed that the shape of meridian is described by the parabola:

$$r(\xi) = r_0\left(1+m\xi^2\right) \tag{4}$$

$$r_0 = \frac{1}{\sqrt{1+\frac{2}{3}m+\frac{m^2}{5}}} \tag{5}$$

where r_0 stands for unknown radius in the middle of the shell, and the directional coefficient m $(m \le 0)$ is a design variable in the optimization procedure.

For a shell with a double non–negative curvature [8] transformed a problem of global stability to a problem of local stability of such a structure. First, the sinusoidal deflection mode was assumed. Next, on the base of the linear theory of shell stability and the equations given by [9], Shirshov obtained formula for the critical loading parameter q, namely:

$$\begin{aligned} q_{1,2} &= 2\sqrt{DEH}\frac{K_\varphi + K_\vartheta z_{1,2}^2}{\overline{N}_\vartheta + 2\overline{S}z_{1,2} + \overline{N}_\varphi z_{1,2}^2} \\ q_{kr} &= \min\left(q_1, q_2\right) \end{aligned} \tag{6}$$

where $z_{1,2}$ are the roots of the quadratic equation:

$$z^2 + \frac{K_\vartheta \overline{N}_\varphi - K_\varphi \overline{N}_\vartheta}{K_\vartheta \overline{S}}z - \frac{K_\varphi}{K_\vartheta} = 0 \tag{7}$$

$$z = tg\phi \tag{8}$$

$$z_{1,2} = -\frac{\overline{N}_\varphi}{\overline{S}} \mp \sqrt{\frac{\overline{N}_\varphi^2}{\overline{S}^2} + \frac{K_\vartheta}{K_\varphi}} \tag{9}$$

In the above equations the following symbols were introduced: K_ϑ and K_φ denote circumferential and meridional curvatures, respectively, $\overline{N}_\vartheta$, $\overline{N}_\varphi$, $\overline{S}$ are the intensity of membrane and shearing forces (due to possible twisting) related to the loading multiplier q, namely $N_\vartheta = q\overline{N}_\vartheta$, $N_\varphi = q\overline{N}_\varphi$, $S = q\overline{S}$, D denotes the shell stiffness, E – Young modulus and ϕ is a certain free parameter with respect to which the loading parameter q is minimized.

The minimization of q with respect to ϕ leads to two solutions: $\phi_1 = 0$ and $\phi_2 = \frac{\pi}{2}$ and finally to very simple formulae for the critical loading multipliers. First one is for a case when the buckling is determined by the circumferential membrane force, whereas second one when the buckling is determined by the meridional membrane force:

$$q_{1\,kr} = 2\sqrt{DEH}\frac{K_\varphi}{\overline{N}_\vartheta} \tag{10}$$

$$q_{2\,kr} = 2\sqrt{DEH}\frac{K_\vartheta}{\overline{N}_\varphi} \tag{11}$$

The critical value of loading multiplier is determined by a smaller one value from (10), (11). For the considered shell $S = 0$, and the membrane forces are as follows:

$$N_\vartheta = \frac{pR_\vartheta}{2}\left(2 - \frac{R_\vartheta}{R_\varphi}\right) \tag{12}$$

$$N_\varphi = \frac{pR_\vartheta}{2} \tag{13}$$

Using equations given above (1), (2), (10), (11) and (12), (13) one can obtain the two equations describing the wall thickness of a shell of uniform stability.

For N_ϑ critical:

$$h_1 = \sqrt{\frac{p_{kr}\sqrt{3\left(1-\nu^2\right)}}{2E}\frac{R_0}{H_0}r_\vartheta\sqrt{(2k-1)}} \tag{14}$$

For N_φ critical:

$$h_2 = \sqrt{\frac{p_{kr}\sqrt{3\left(1-\nu^2\right)}}{2E}\frac{R_0}{H_0}r_\vartheta} \tag{15}$$

For a shell of a medium length considered in this paper, the critical component of membrane force is circumferential one, so the equation (14) holds and $q_{kr} = q_{1\,kr}$.

3. Formulation of the optimization problem

The optimization task is formulated as a nonlinear programming problem. In first stage of the optimization with given parabolic shape of a meridian, we look for such a value of parabola directional coefficient and distribution of a wall thickness of the shell, which lead to the maximal value of the critical loading multiplier:

$$q_{kr} \to \max \tag{16}$$

Such an optimization problem is stated under two inequality and one equality constraints. It is assumed that an optimal shell is made with no more amount of material than a cylindrical reference shell,

$$2\pi L_0 R_0 H_0 \geq 2\pi \int_0^{L_0} H R_\vartheta dx \tag{17}$$

the internal capacity of both containers is equal:

$$2\pi L_0 R_0^2 = 2\pi \int_0^{L_0} R^2 dx \tag{18}$$

Moreover, the minimal value of the coordinate R which occurs at the interface of the coat and bottom of a shell is constrained by a lower bound,

$$R(L_0) = R_{\min} \geq R_{adm} \tag{19}$$

where R_{adm} is an arbitrary chosen value. The measure of the optimal shell profit is ratio of critical pressure of the optimal shell to the critical pressure of the reference cylinder, that is the critical value of a loading multiplier,

$$q_{kr} = \frac{p_{kr}}{p_{kr}^{cyl}} \tag{20}$$

where p_{kr}^{cyl} is defined by the following formula:

$$p_{kr}^{cyl} = \frac{\pi\sqrt{6}E}{18\left(1-\nu^2\right)^{3/4}} \frac{R_0}{L_0} \left(\frac{H_0}{L_0}\right)^{5/2} \tag{21}$$

which can be found in e.g. [10].

In next stage of the optimization, the optimal shell from first stage is reinforced with additional ribs. The assumed number of ribs equals three and is kept constant. The ribs are modeled as thin-walled plates. It is assumed that an every rib has a hole of arbitrary chosen radius value, equals 92 % of the shell radius in the place where the rib is welded to the coat of a shell. The design variables are the positions of ribs in the shell:

$$\zeta_i \quad i = 1, 2, 3 \tag{22}$$

which are measured from the origin of the coordinate system. These design variables are treated as integer ones due to the density of finite element mesh along the meridian of the shell and have to be in a range $\zeta_i \in [0, 100]$. It is obvious that the ribs stiffen the shell and the critical value of loading multiplier should be greater. The objective function is the same as in the first stage of the optimization (16). The shape of meridian is kept unchanged so the constraints (18) and (19) are automatically fulfilled (it is assumed that the capacity of a shell does not depend on a volume of ribs). By contrast, the amount of material has to be distributed on the coat of the shell and its ribs:

$$2\pi L_0 R_0 H_0 \geq \alpha \cdot 2\pi \int_0^{L_0} H R_\vartheta dx + \sum_{i=1}^{3} V_i \quad i = 1..3 \tag{23}$$

where V_i is a volume of i–th rib, which a constant thickness equals a thickness of shell where a rib is welded to a coat. The factor $\alpha \in [0.9,\, 1]$ can be named as a thinning coefficient of a shell of uniform stability.

4. Method of solution

The Finite Element Method (FEM) was used to solve the analysis task. The parametric model of the shell was built in APDL (ANSYS Parametric Design Language) in ANSYS software. The formula (14) defining the wall thickness of shell of uniform stability was coded in script language what made it possible to carry out calculations in first stage of the optimization. The results of analysis are passed to the Modified Particle Swarm Optimization (MPSO) algorithm.

The PSO algorithm is inspired by social living forms: bee swarms, bird flocks and fish schools from the world of the nature. Individuals forming a swarm influence each other and are affected by the environment, simultaneously. A population of particles which are understood as points in multidimensional space is initialized with random positions and velocities. They are updated then at each time step while flowing over the search space. The velocity vector is updated based on its own memory and information gained by the swarm, for each particle. The position is updated based on the previous position and velocity vectors of each particle. The update equations of the moving swarm MPSO algorithm are expressed below

$$\mathbf{v}_{k+1}^{i} \leftarrow w_1 \mathbf{v}_k^i + w_{2k}^i \left[c_1 r_{1k}^i \left(\mathbf{p}_k^{\mathbf{i}i} - \mathbf{x}_k^i \right) + c_2 r_{2k}^i \left(\mathbf{p}_k^{\mathbf{g}} - \mathbf{x}_k^i \right) + c_3 r_{3k}^i \left(\mathbf{p}_k^{\mathbf{n}i} - \mathbf{x}_k^i \right) \right] \tag{24}$$

$$\mathbf{x}_{k+1}^{i} \leftarrow \mathbf{x}_k^i + \mathbf{v}_{k+1}^i \tag{25}$$

where the following symbols are applied:

k – iteration step index,
i – particle index,
c_1, c_2, c_3 – fixed coefficients named as acceleration constants or learning factors,
r_1, r_2, r_3 – uniformly distributed random numbers in range [0,1],
w_1 – inertia weight, w_2 – binary switching coefficient,
$\mathbf{x}$ – position vector,
$\mathbf{v}$ – velocity vector,
$\mathbf{p^i}$ – the best own particle position found so far,
$\mathbf{p^n}$ – the best particle neighbours leader position found so far,
$\mathbf{p^g}$ – the best swarm leader position found so far.

The dimension of the position and velocity vectors equals the number of design variables in the optimization task. The fitness function evaluates the particle position by calling the FEM analysis procedure. The MPSO deals with nonlinear programming task with equality and inequality constraints. The "cut–off" at the boundary technique was applied for constraints handling. The work of the particle swarm algorithm is managed by just a few parameters. Nevertheless, choosing the best value for these is crucial for obtaining a rapid solution and a correct result. It is highly probable that the solution obtained by the MPSO algorithm is a global optimum.

The optimal shell was analysed using FEM in linear buckling analysis in ANSYS software to determine the buckling load (first eigenvalue) and shape of its mode. The eigenvalue problem is formulated in the following manner,

$$(\mathbf{K} + \lambda_i \mathbf{S})\ \psi_i = \mathbf{0} \tag{26}$$

where the following symbols are applied:

K – a stiffness matrix,
S – a stress stiffness matrix (built in a static solution with prestress effects activated),
λ_i – i–th eigenvalue,
ψ_i – i–th eigenvector of displacements (mode shape).

The pressure is considered as a follower load so the force on the surface is a function of a pressure value and an orientation of a surface. The calculated multipliers λ_i are equal buckling loads if a unit load is specified. In this case a value of critical pressure equals a value of first eigenvalue:

$$p_{kr} = \lambda_1 \tag{27}$$

From formulas (20) and (27) it is easy to obtain the relation between critical loading multiplier and first eigenvalue from buckling analysis:

$$q_{kr} = \frac{\lambda_1}{p_{kr}^{cyl}} \tag{28}$$

5. Numerical results

The geometrical and material data of the shell are as following: longitudinal parameter $\mu = 0.25$, the wall thickness of the reference shell $H_0 = 0.005R_0$ [m], the Young modulus $E = 2.1e5$ [MPa], Poisson ratio $\nu = 0.3$, the radius of the reference shell $R_0 = 1$ [m] and $R_{adm} = 0.5$ [m]. The swarm was constituted by three particles. Every one particle had one neighbour. The stop criterion of optimization was number of iteration steps equals 25.

The history of the optimization profit and parabola directional coefficient on subsequent steps is presented in Fig. 2 for first stage of the optimization. In Fig. 3 the history of second stage of the optimization is presented. The results of optimization are given in Tab. 1.

Table 1 The values of critical loading multipliers, design variables and material constraint for the optimal shells

	Shell not stiffened		Shell stiffened by ribs
	Uniform stability	Buckling analysis	Buckling analysis
Critical loading multiplier q_{kr}	6.595152	6.634736	8.926179
Design variables	m = -0.588637		$\zeta_1 = 45$ $\zeta_2 = 76$ $\zeta_3 = 14$ $\alpha = 0.926061$
Constraint value [m^3]	0.125663		0.123775
Limit value of constraint [m^3]	0.125663		

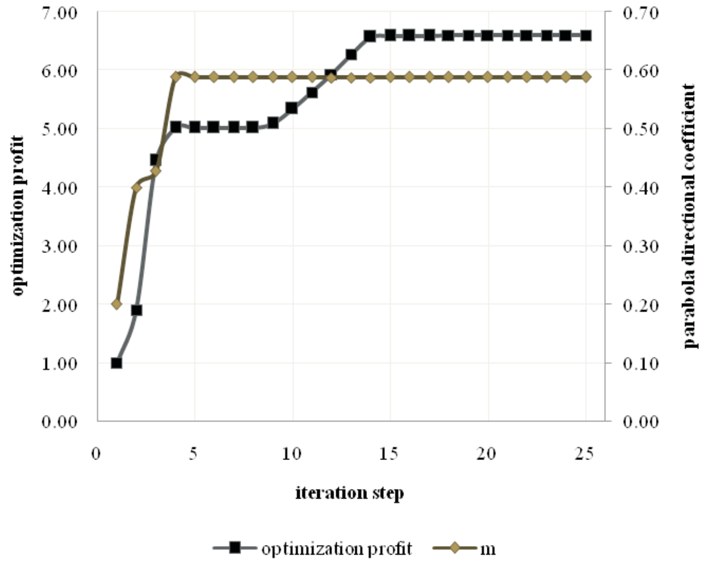

Figure 2 The history of first stage of the optimization

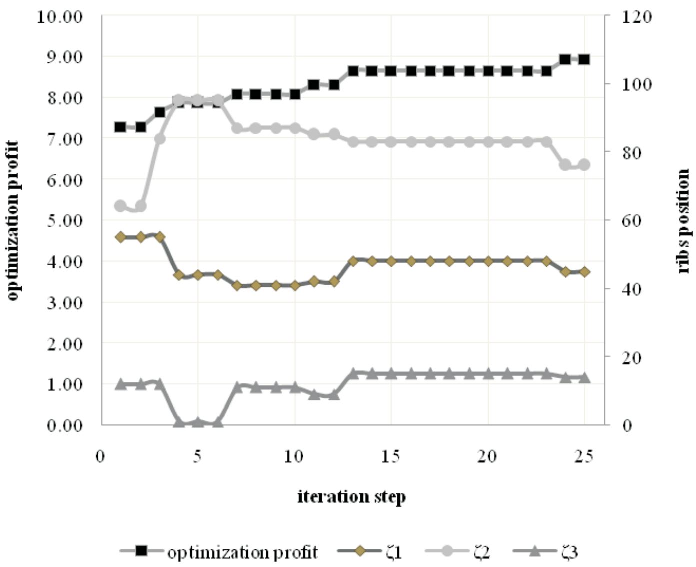

Figure 3 The history of second stage of the optimization

In both stages of the optimization the active constraint was amount of material, given by formula (17) in first stage or (23) in second one. The obtained results confirm these known in literature, e.g. [2, 11] in which optimization task was solved by parametrical optimization. The value of critical loading multiplier was verified by buckling analysis in ANSYS. The results are similar each other, so one can concludes that this value is positively verified.

Fig. 4 shows the shape of a meridian and distribution of the wall thickness of the optimal shell obtained in the first stage of optimization. The reference structure is shown in the background, also (the bottoms are not shown). The wall thickness was multiplied by 30 for better presentation, whereas radii kept its original values. It can be observed that the wall thickness distribution is a decreasing function.

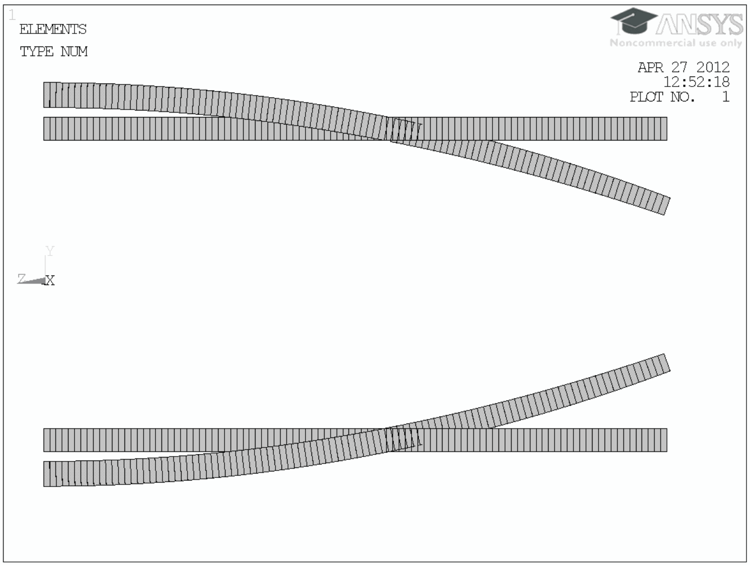

Figure 4 The wall thickness distribution and a shape of meridian of the optimal shell without ribs (the coat of shell is shown only)

In Fig. 5 the shape of the first mode of buckling for the optimal shell obtained in the first stage is presented.

In the second stage of optimization the arrangements of ribs as well as the value of thinning coefficient of shell are looked for. It was successful to get the increase of critical multiplier value. The shape of the first mode of buckling for the optimal shell is presented in Fig. 6.

It is supposed to obtain even better results if the number of ribs, the hole radius in rib, and wall thickness distribution (constant or variable) in every rib would be treated as the design variables.

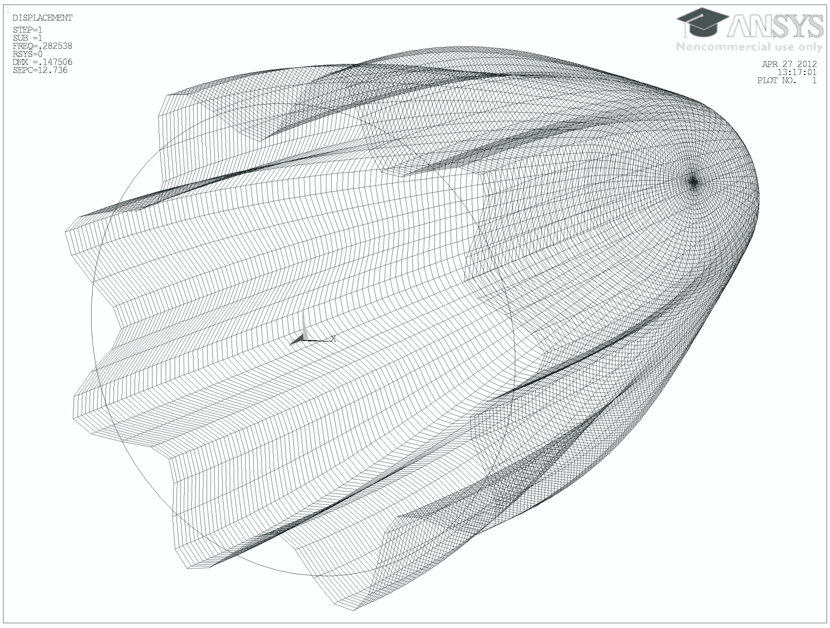

Figure 5 The shape of first buckling mode of the optimal shell without ribs

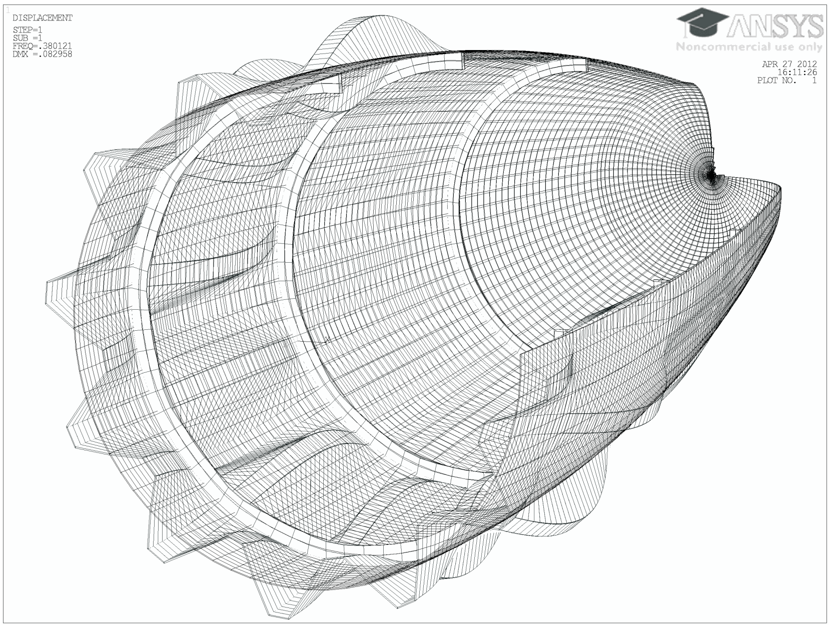

Figure 6 The shape of first buckling mode of the optimal shell stiffened by ribs (the shell fragment was removed to improve readability)

6. Concluding remarks

On the basis of the above calculations it was found that for the shell of revolution, medium long, significantly increase of the critical loading multiplier can be obtained by determining the optimal shape of the meridian, the optimal distribution of wall thickness in coat of shell and – which gives a substantial profit – optimal placement of ribs inside the shell. The results of optimization in formulation of uniform stability were successfully verified by numerical buckling analysis. The applied Modified Particle Swarm Optimization algorithm gives global optimal results in nonlinear optimization tasks with constraints for the continuous and integer type design variables.

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