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Vibrations and Stability of a Column Subjected to the Specific Load Realized by Circular Elements of Heads

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Research into columns subjected to a specific load considering a constructional solution consisting of circular elements is an aim of the paper. The boundary problem of stability and free vibrations of the considered system for a generalised load by a force directed towards the positive pole was formulated on the basis of Hamilton's principle. The boundary conditions concerning the load by a follower force directed towards the positive pole were deduced on the basis of the boundary conditions of the column subjected to a generalised load by a force directed towards the positive pole. In this paper, the load by the follower force was treated as a special case of the generalised load by a force directed towards the positive pole. Apart from the formulation of and solution to the boundary problem of the columns subjected to the specific load, the results of numerical computations connected to the static problem and free vibrations of the considered systems were also presented.

Keywords: Slender system, buckling, divergence instability, columns.

1. Introduction

In relevant literature one can find many works dedicated to slender systems. Different types of loads, both a conservative and non-conservative, were studied by authors of these works. Euler's load [1, 2, 3, 4], generalised load [5, 6, 7] and a specific load [8–10, 11–14, 15–17] can be classed as conservative loads while Beck's load (load by a follower force), generalised Beck's load (with follower load factor) and loads coupled with them, that is Reut's load and generalised Reut's load, can be classes as non-conservative loads [18, 19, 20, 21, 22]. Further part of introduction was limited to the specific load which is a topic of considerations carried out in the frame of this elaboration. The conservative specific load was formulated and introduced by L. Tomski (comp. [8]) can be realised through loading structures made of curvilinear elements (characterised by circular contour [10–12, 14, 15] or parabolic contour [16] or linear elements [8–10, 13, 17, 23]. Two basic types of a specific load can be distinguished: generalised load with the force directed towards the pole (positive or negative) [8, 9, 11, 13, 15, 17], and a load generated by the follower force directed towards the pole (positive or negative) [10, 12–14, 23]. Letter denotations of these two basic types of a specific load are introduced in the further part of the paper. Denotation S_G corresponds to generalised load by a force directed towards the positive pole whereas denotation S_F is connected to the load by the follower force directed towards the positive pole. The positive pole is located below the loaded end of the column. The specific loads S_G and S_F are characterised by the other boundary conditions. In the case of the system S_G, two natural boundary conditions are distinguished at the loaded end of this system and in the case of the system S_F geometrical and natural boundary condition is present. Deflection is connected to deflection angle of the column through the geometrical boundary condition for the load S_F.

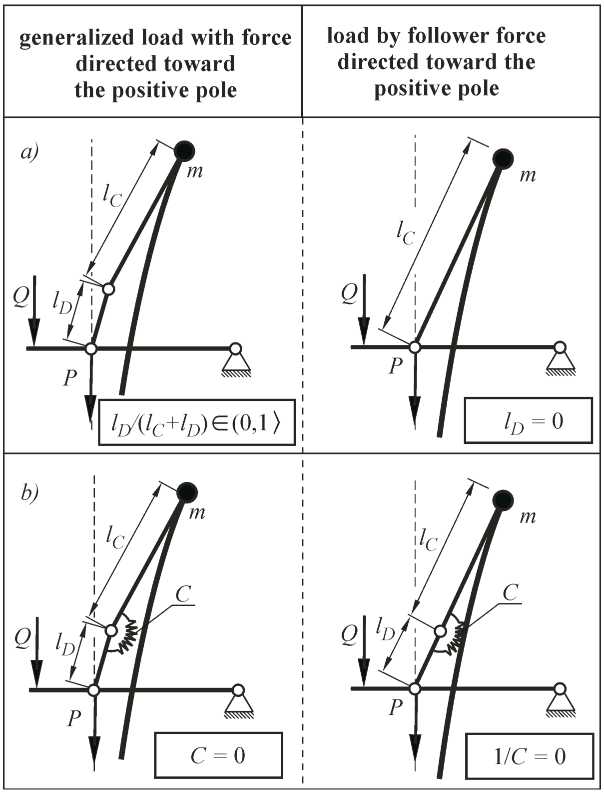

Figure 1 A specific load considering technical realisation consisting of linear elements: a) without a rotational spring between rigid bolts, b) with a rotational spring between rigid bolts

Loading structure (built of linear elements) realising the specific load $S_G$consists of two rigid bolts of l_D and l_C in length (Fig. 1), respectively. Such a construction of loading heads makes it possible to obtain the load S_F on the basis of the load S_G in two ways. The length of element l_D must be assumed as equal zero to obtain the load S_F in the first method (Fig. 1a). In the second method (Fig. 1b), which was considered in work [13], a rotational spring between bolts of l_D and l_C in length was taken into account. In work [13] it was proved that for heads built of linear elements the load S_F is a special case of the load S_G (when applying infinitely high rigidity of a rotational spring).

An aim of this paper is research into column subjected to the load S_G considering also its special case that is the load S_F. The construction, built of circular elements , is taken into account in this work too (contrary to work [13]).

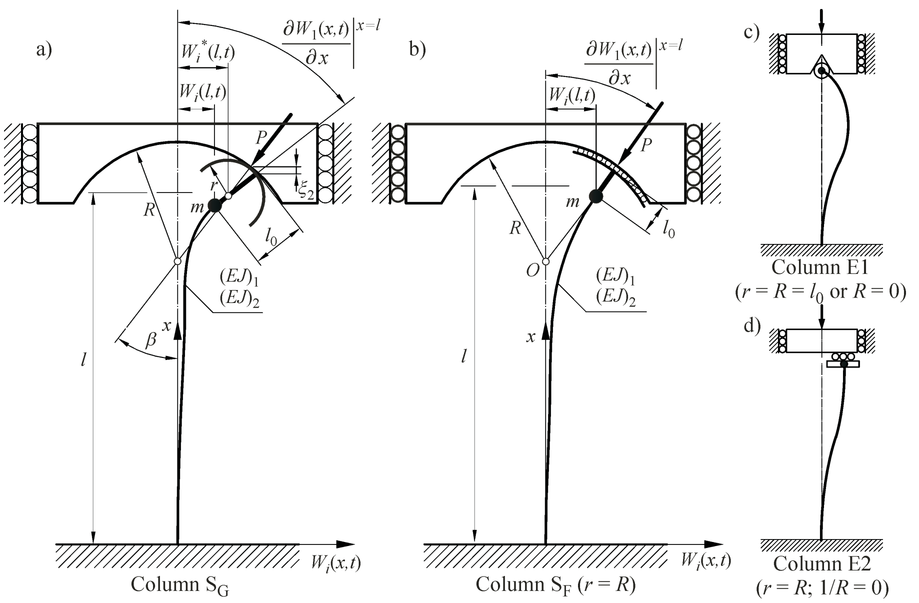

Figure 2 Columns with the loading systems: a) the column subjected to generalised load by a force directed towards the positive pole, b) the column subjected to a load by the follower force directed towards the positive pole, c), d) the column subjected to the Euler's load

2. The Formulation of the Boundary Problem of the Considered System

The considered slender system (column) is presented in Fig. 2. The column is built as a flat frame consisting of two identical rods symmetrically placed towards the axis of the system. In this case a specific load is generated by the loading heads made of circular elements. The column subjected to the generalised load by a force directed towards the positive pole is presented in Fig. 2a. Loading system consists of loading head (characterised by radius of curvature R) and receiving head (characterised by radius of curvature r). Rods of the column with rolling element of radius r were connected by the rigid bolts of $l_0$in length. The boundary case of the load S_G for $r = R$ (Fig. 2b), which is the load by the follower force directed towards the positive pole (the load S_F), is presented in Fig. 2. The remaining special cases of the load S_G are presented in Fig. 2c (the column denoted by E1 ($l_0 = R$)) and Fig. 2d (the column denoted by E2 ($r = R$ and $1/R = 0$)).

The following denotations were accepted regarding the geometrical and physical quantities of the considered systems: $W_i(x,t)$ – transversal displacement of the column rods corresponding to coordinate x and time t, $(EJ)_i$– bending rigidity of the i – th rod of the column, $(\rho_0 A)_i$ – mass per unit length of the column rod, P – external force loading the column, ω - the natural frequency.

At the ends of the considered column (unbiased ($x = 0$) and loaded ($x = l$)) the geometrical boundary conditions are as follows:

$$W_1\left(0,t\right) = W_2\left(0,t\right) = \left.\frac{\partial W_1\left(x,t\right)}{\partial x}\right|_{x=0} = \left.\frac{\partial W_2\left(x,t\right)}{\partial x}\right|_{x=0} = 0, \tag{1}$$

$$W_1\left(l,t\right) = W_2\left(l,t\right), \qquad \left.\frac{\partial W_1\left(x,t\right)}{\partial x}\right|^{x=l} = \left.\frac{\partial W_2\left(x,t\right)}{\partial x}\right|^{x=l} \tag{2}$$

In the case of taking into account a special case of the load S_G it means the load S_F (Fig. 2b) there is one more boundary condition present in the system, by which the bending of loaded end of the column is dependent on the bending angle of this end. This geometrical condition can be presented as follows:

$$\psi = \left.\frac{\partial W_1\left(x,t\right)}{\partial x}\right|^{x=l}\left(R - l_0\right) - W_1\left(l,t\right) = 0 \tag{3}$$

In this work the boundary problem of the considered systems is deduced on the basis of Hamilton's principle taking into account the load of S_G type. The boundary conditions in relation to S_F load (for $r = R$) are determined on the basis of the boundary problem.

For the conservative systems the Hamilton's principle is in the following form:

$$\delta\int\limits_{t_1}^{t_2}\left(T - V\right)dt = 0 \tag{4}$$

Energies: the kinetic T and the potential V of the system subjected to the load (Fig. 2a) are as follows:

$$T = \frac{1}{2}\sum_{i=1}^{2}\int\limits_0^l\left(\rho_0 A\right)_i\left[\frac{\partial W_i\left(x,t\right)}{\partial t}\right]^2 dx + \frac{1}{2}m\left[\left.\frac{\partial W_i\left(x,t\right)}{\partial t}\right|^{x=l}\right]^2 \tag{5}$$

$$\begin{gathered}V = \tfrac{1}{2}\sum\limits_{i=1}^{2}\left(EJ\right)_i\int\limits_0^l\left[\tfrac{\partial^2 W_i(x,t)}{\partial x^2}\right]^2 dx - P\tfrac{1}{2}\int\limits_0^l\left[\tfrac{\partial W_1(x,t)}{\partial x}\right]^2 dx - \\ \tfrac{1}{2}Pl_0\left[\left.\tfrac{\partial W_1(x,t)}{\partial x}\right|^{x=l}\right]^2 + \tfrac{1}{2}Pr\left\{\left[\left.\tfrac{\partial W_1(x,t)}{\partial x}\right|^{x=l}\right]^2 - \gamma^2\right\} + \tfrac{1}{2}P\gamma W^*\end{gathered} \tag{6}$$

where:

$$\gamma = \frac{1}{(R-r)}\left(W_1\left(l,t\right) + \left(l_0 - r\right)\left.\frac{\partial W_1\left(x,t\right)}{\partial x}\right|^{x=l}\right) \tag{7}$$

$$W^* = W_1\left(l,t\right) + \left.\frac{\partial W_1\left(x,t\right)}{\partial x}\right|^{x=l}\left(l_0 - r\right) + r\gamma \tag{8}$$

By substituting the energies written by equations (5) and (6) into Hamilton's principle and applying the geometrical boundary conditions (5) one can obtain:

– the differential equation of the motion in transversal direction:

$$\left(EJ\right)_i\frac{\partial^4 W_i\left(x,t\right)}{\partial x^4} + \frac{P}{2}\frac{\partial^2 W_i\left(x,t\right)}{\partial x^2} + \left(\rho A\right)_i\frac{\partial^2 W_i\left(x,t\right)}{\partial t^2} = 0 \tag{9}$$

– natural boundary conditions:

$$\sum_{i=1}^{2}\left(EJ\right)_i\left.\frac{\partial^2 W_i\left(x,t\right)}{\partial x^2}\right|^{x=l} + P\frac{r-l_0}{R-r}\left[\psi\right] = 0 \tag{10}$$

$$\sum_{i=1}^{2}\left(EJ\right)_i\left.\frac{\partial^3 W_i\left(x,t\right)}{\partial x^3}\right|^{x=l} + P\frac{1}{R-r}\left[\psi\right] - m\left.\frac{\partial^2 W_1\left(x,t\right)}{\partial t^2}\right|^{x=l} = 0 \tag{11}$$

There is no necessity to deduce the boundary conditions corresponding to the load F on the basis of Hamilton's principle. The boundary conditions present for the load can be derived on the basis of conditions (8) because the load S_F is a special case of the load S_G. Considering the load S_F (when $r = R$ (Fig. 2b)) the condition (3) is taken into account. The quantity ψ (3), dependent on bending and bending angle of the loaded end, is also present in the derived boundary conditions in relation to the load S_G (conditions (10) and (11)). To obtain the natural boundary condition connected to the load S_F , one must determine the quantity ψ from the equation (10) and then substitute it into the equation (11). The natural boundary condition for the load by the follower force directed towards the positive pole is written as:

$$\begin{aligned}&\sum_{i=1}^{2}\left(EJ\right)_i\left.\frac{\partial^3 W_i\left(x,t\right)}{\partial x^3}\right|^{x=l} - \frac{1}{R-l_0}\sum_{i=1}^{2}\left(EJ\right)_i\left.\frac{\partial^2 W_i\left(x,t\right)}{\partial x^2}\right|^{x=l}\\&-m\left.\frac{\partial^2 W_1\left(x,t\right)}{\partial t^2}\right|^{x=l} = 0\end{aligned} \tag{12}$$

3. Solution to the Boundary Problem

A harmonic solution to the differential equations (9) is accepted:

$$W_i\left(x,t\right) = y_i\left(x\right)\cos\left(\omega\, t\right) \tag{13}$$

Considering solution (13), the differential equations of motion in transversal direction are modified to the form:

$$\left(EJ\right)_i\frac{d^4 y_i\left(x\right)}{dx^4} + \frac{P}{2}\frac{d^2 y_i\left(x\right)}{dx^2} - \left(\rho A\right)_i\omega^2 y_i\left(x\right) = 0 \tag{14}$$

The boundary conditions corresponding to adequate case of load S_G or S_F can be written in the similar way.

The solution of differential equations (14) are functions:

$$y_i(x) = D_{i1}\cosh(\alpha_{i1}x) + D_{i2}\sinh(\alpha_{i1}x) + D_{i3}\cos(\alpha_{i2}x) + D_{i4}\sin(\alpha_{i2}x) \tag{15}$$

where:

$$\alpha_{ij} = \sqrt{(-1)^j \frac{1}{2}\frac{P}{(EJ)_i} + \sqrt{\frac{1}{4}\left(\frac{P}{(EJ)_i}\right)^2 + \frac{\omega^2(\rho A)_i}{(EJ)_i}}} \tag{16}$$

Substitution of equation (15) into the boundary conditions, after separation of variables, leads into the following system of equations:

$$[a_{ij}]\, col\{D_{11}, D_{12}, D_{13}, D_{14}, D_{21}, D_{22}, D_{23}, D_{24}\} = 0 \tag{17}$$

The determinant of coefficients matrix of this system is the transcendental equation for the natural frequency of the column ω:

$$|a_{ij}| = 0 \tag{18}$$

4. The Results of Numerical Computations

The results of numerical computations are presented with regard to the dimensionless quantities defined in the following way:

$$\zeta_A = \frac{R}{l}; \quad \zeta_B = \frac{r}{R}; \quad \zeta_C = \frac{l_0}{R}; \quad \zeta_D = \frac{m}{\sum_i (\rho A)_i l} \tag{19}$$

$$\lambda_{cr} = \frac{P_{cr}l^2}{\sum_i (EJ)_i}; \quad \lambda = \frac{Pl^2}{\sum_i (EJ)_i}; \quad \Omega = \frac{\omega^2 \sum_i (\rho A)_i l^4}{\sum_i (EJ)_i} \tag{20}$$

Dimensionless parameter of the critical load λ_{cr} in relation to parameter of the loading head radius ζ_A is presented in Figs. 3 and 4.

Numerical computations (corresponding to the column S_G) were carried out for different values of parameter ζ_B and $\zeta_C = 0.5$. Their results are presented in Fig. 3.

The results of numerical computations in relation to the system S_F ($\zeta_B = 1.0$ t Fig. 4) are presented for different values of parameter ζ_C. In Figs. 3 and 4, curves corresponding to the special cases of the considered loads are denoted by different type of line and by point A. The special cases of the column S_G are as follows: column S_F (Fig. 2b ($r = R \Leftrightarrow \zeta_B = 1$)), column E1 (Fig. 2c ($R = 0 \Leftrightarrow \zeta_A = 0$)). The special cases of the column S_F are as follows: column E1 (Fig. 2c ($R = 0 \Leftrightarrow \zeta_A = 0$) and ($l_0 = R \quad \Leftrightarrow \zeta_C = 1.0$)), column E2 (Fig. 2d ($1/R = 0 \Leftrightarrow 1/\zeta_A = 0$)).

The column S_G is characterised by the highest value of the critical load if $r = R(\zeta_B = 1.0)$ what corresponds to the column S_F (Fig. 3 t dashed curve). The parameters of the considered systems can be selected in such a way that the highest critical load can be obtained.

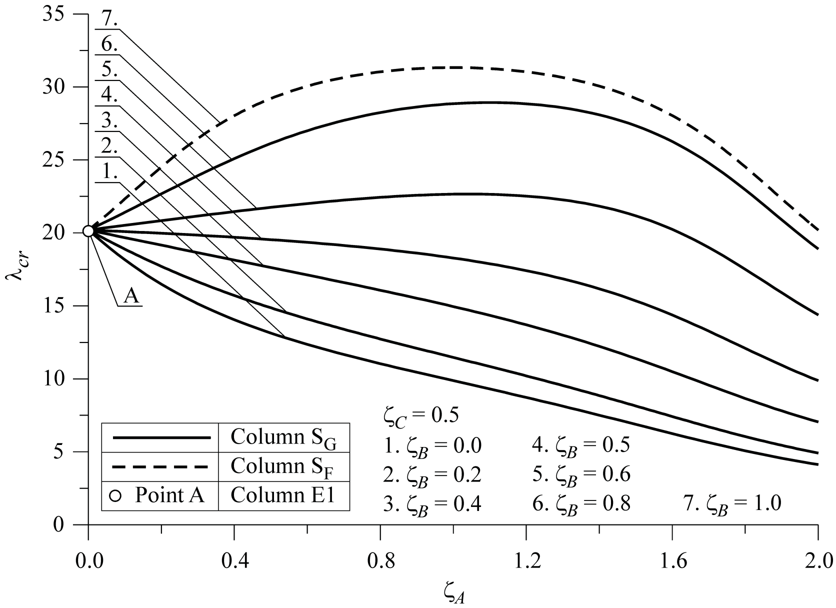

Figure 3 Parameter of the critical load λ_{cr} in relation to parameter ζ_A for different values of parameter ζ_B(ζ_B = 0, 0.2, 0.4, 0.5, 0.6, 0.8, 1.0) and for $\zeta_C = 0.5$

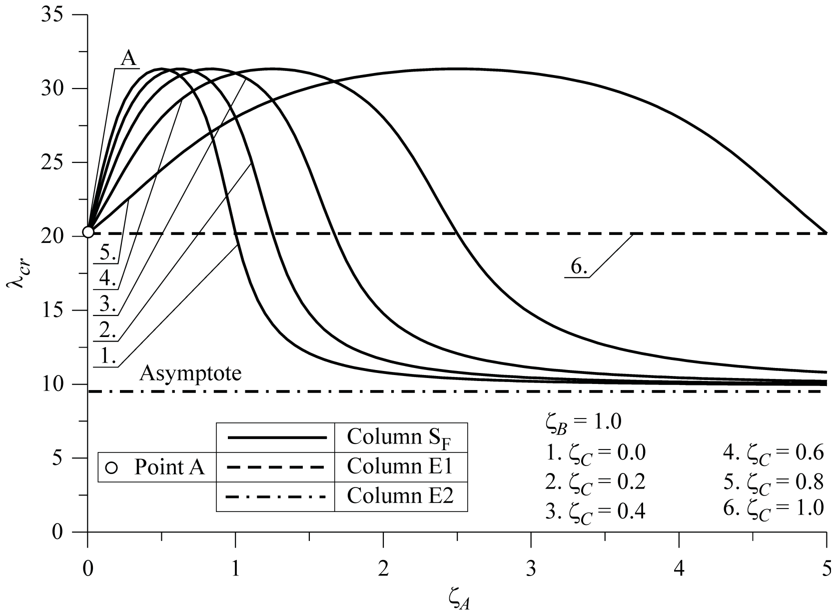

Figure 4 Parameter of the critical load λ_{cr} in relation to parameter ζ_A for different values of parameter ζ_C (ζ_C = 0, 0.2, 0.4, 0.6, 0.8, 1.0) and for $\zeta_B = 1.0$

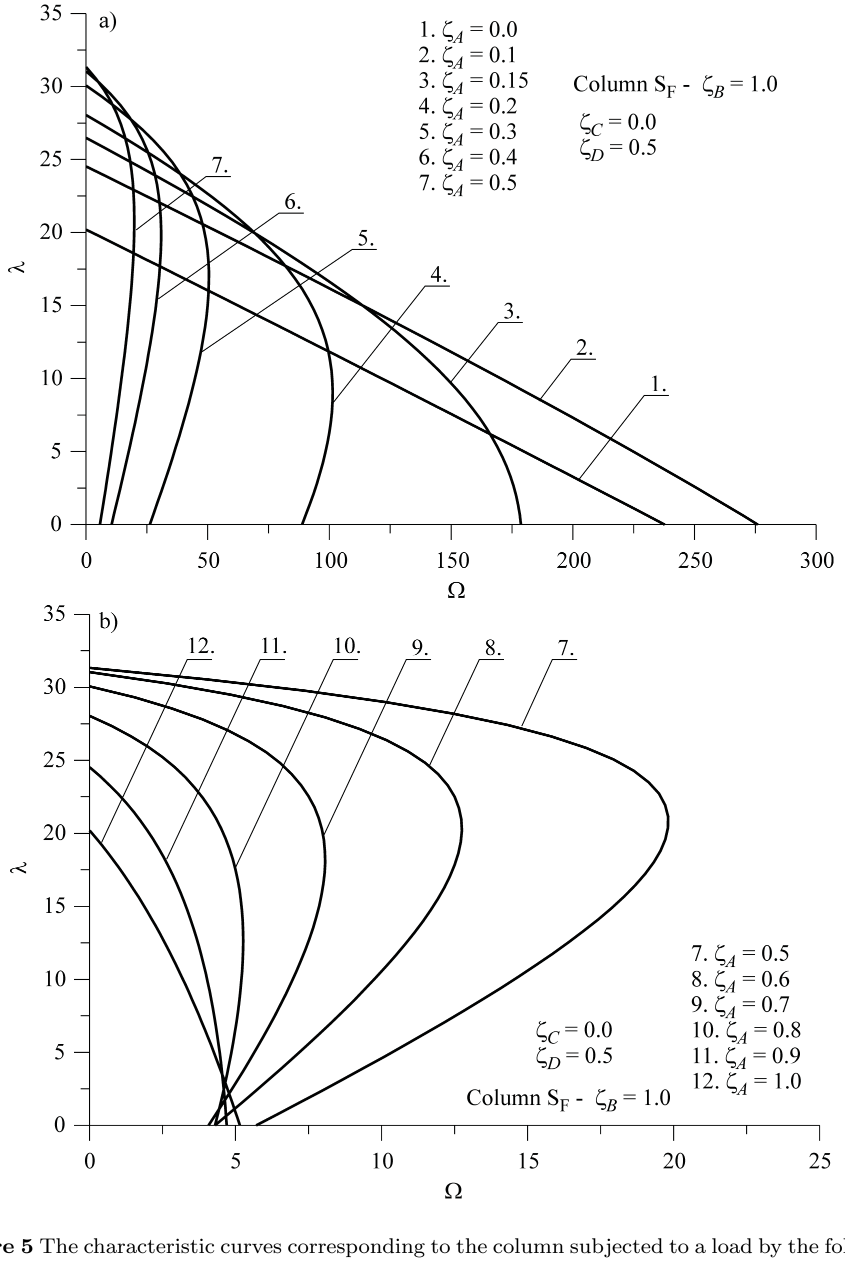

Figure 5 The characteristic curves corresponding to the column subjected to a load by the follower force directed towards the positive pole

The critical load of the considered systems was determined on the basis of the kinetic criterion of stability. To that end the characteristic curves in the plane: load natural frequency were determined. Taking into account the characteristic curves, the critical load corresponds to zero value of natural frequency. Exemplary characteristic curves corresponding to the system S_F are presented in Figs. 5a i 5b. Numerical computations were carried out for different values of the parameter ζ_A (Fig. 5a – $\zeta_A \in \langle 0{,}0.5 \rangle$, Fg. 5b – $\zeta_A \in \langle 0.5{,}1.0 \rangle$) and $\zeta_C = 0$, $\zeta_D = 0.5$. In the case of a specific load (such the load is considered in this work) two types of characteristic curves can be obtained which differ in slope angle for the zero value of external load. If the slope is negative, the characteristic curves are of divergence type. The characteristic curves corresponding to positive slope for the zero value of external load are of divergence pseudo-flutter type. The divergence curves are present for lower and higher value of the parameter ζ_A ($\zeta_A = 0$, 0.1, 0.15, 0.9, 1.0).

5. Conclusion

A geometrically linear slender system subjected to a specific load: generalised by a force directed towards the positive pole and by a follower force directed towards the positive pole was considered in the paper. The specific load was realised by loading systems built of circular elements. The boundary problem of free vibrations for a generalised load by a force directed towards the positive pole was formulated on the basis of Hamilton's principle. The load by the follower force directed towards the positive pole was treated as a special case of the generalised load by a force directed towards the positive pole. The critical load of the considered systems was determined on the basis of the kinetic criterion of stability. Special cases of the tested columns, present for the boundary values of the parameters of the loading systems, were given in this paper. The results of numerical computations connected to the critical load and characteristic curves were carried out for different values of the parameters of the systems.

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