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Effect of Galactic Rotation on Radial Velocities and Proper Motion Part II

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We express the geometrical and algebraic aspects of the problem of galactic rotation on the motion of the stars represented by fig. (2). We verify the equations involving third order terms of the orbits of the stars. That means taking into account higher order terms in our analysis, namely up to $O(r/R_0)^3$. These terms allow a generalization and high precision for the results. We acquired a higher order Taylor's expansion for V as denoted in fig. (2). U‘, V‘ are the linear components of the velocity of the group of stars S. After some lengthy expansions and reductions, we obtained the formulae for U‘, V‘. Consequently ξ, η, ζ the linear components of S corresponding to the two proper motion equalities in galactic longitude and latitude and radial velocity $\Delta\rho$. Expansions are performed up to the third order in (r/R_0).

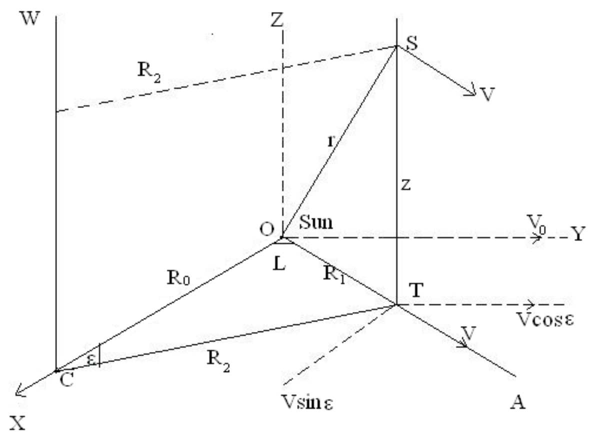
Keywords: Orbital mechanics of the stars, stellar kinematics, galactic dynamics

1. Introduction

We devote the introduction as in Part I, to the assumptions, of the following Fig. (2).

(i) S denotes a group of stars at mean distance r from Sun, and z from galactic plane. X is in the direction of the galactic center C (long. G_0); Y axis in long. $(90+G_0)$; Z perpendicular to galactic plane.

(ii) S is in circular velocity, for the time being, V about the axis CW parallel to OZ and parallel to galactic plane. The distance from S to CW is denoted by R_2; T is the projection of S on XY plane, CT=R_2.



(iii) V could be represented by a vector along TA, with components V sinε and V cosε parallel to OX and OY respectively, ε is the angle OCT, L=G-G$_0$, G$_0$ the galactic long. of C the center.

2. Equations with third order terms for the orbits of the Stars

Relative to O, the linear components of S are U‘ parallel to OX and V‘ parallel to OY, where

$$U' = V \sin \varepsilon \tag{1}$$

$$V' = V \cos \varepsilon - V_0 \tag{2}$$

The component W‘= 0.
It could be proved that

$$\begin{aligned} -U' \sin L + V' \cos L &= \xi \\ -U' \cos L \, \sin g - V' \sin L \, \sin g &= \eta \\ U' \cos L \, \cos g + V' \sin L \, \cos g &= \zeta \end{aligned} \tag{3}$$

Where ξ, η, ζ are the linear components of S corresponding to the proper motion components $\Delta\mu_G \cos g$, $\Delta\mu_g$ and $\Delta\rho$ the radial component respectively; G, g the galactic coordinates of S, more specifically g is the galactic latitude of the group of stars S.

Also from spherical astronomy it could be proved that

$$\xi = \frac{k}{p}\Delta\mu_G \cos g \qquad \eta = \frac{k}{p}\Delta\mu_g \qquad \zeta = \Delta\rho \tag{4}$$

The distance r in parsecs is $1lp$ where $p$is the parallax in seconds of arc; whence

$$\xi = kr\Delta\mu_G \cos g \qquad \eta = kr\Delta\mu_g \qquad \zeta = \Delta\rho \tag{5}$$

The general expansion of $V = V(R_2, z)$, putting

$$R_2 = R_0 + \Delta R = R \tag{6}$$

$$\begin{aligned} V &= V_0 + \Delta R\left(\frac{\partial V}{\partial R}\right) + z\left(\frac{\partial V}{\partial z}\right) + \frac{1}{2}(\Delta R)^2\left(\frac{\partial^2 V}{\partial R^2}\right) + z\Delta R\left(\frac{\partial^2 V}{\partial R\partial z}\right) \\ &+\frac{1}{2}z^2\left(\frac{\partial^2 V}{\partial z^2}\right) + \frac{1}{6}(\Delta R)^3\left(\frac{\partial^3 V}{\partial R^3}\right) + \frac{1}{2}z(\Delta R)^2\left(\frac{\partial^3 V}{\partial R^2\partial z}\right) \\ &+\frac{1}{2}z^2(\Delta R)\left(\frac{\partial^3 V}{\partial R\partial z^2}\right) + \frac{1}{6}z^3\left(\frac{\partial^3 V}{\partial z^3}\right) \end{aligned} \tag{7}$$

The differential coefficients being evaluated at O i.e. $R = R_0$ and z=0.
For symmetry to the galactic equator

$$\frac{\partial V}{\partial z} = \frac{\partial^2 V}{\partial R\partial z} = 0 \tag{8}$$

Thus

$$V = V_0 + a\Delta R + b(\Delta R)^2 + cz^2 \tag{9}$$

up to the second order of smallness, where

$$a = \left(\frac{\partial V}{\partial R}\right) \qquad b = \frac{1}{2}\left(\frac{\partial^2 V}{\partial R^2}\right) \qquad c = \frac{1}{2}\left(\frac{\partial^2 V}{\partial z^2}\right) \tag{10}$$

Now

$$R = R_2 = R_0 + \Delta R = \left(R_0^2 - 2R_0R_1\cos L + R_1^2\right)^{1/2} \tag{11}$$

Also, we have

$$R_1 = r\cos g \qquad z = r\sin g \tag{12}$$

To the third order in r we find that

$$\begin{aligned} \Delta R = R_0 \Bigg[& -\frac{r}{R_0}\cos L + \left(\frac{r}{R_0}\right)^2\left\{\frac{1}{4}(1-\cos 2L)\right\} \\ & + \left(\frac{r}{R_0}\right)^3\left\{\frac{1}{8}(\cos L - \cos 3L)\right\}\Bigg] \end{aligned} \tag{13}$$

Whence by squaring

$$(\Delta R)^2 = R_0^2\left[\left(\frac{r}{R_0}\right)^2\left\{\frac{1}{2}(1+\cos 2L)\right\} - \left(\frac{r}{R_0}\right)^3\left\{\frac{1}{4}(\cos L - \cos 3L)\right\}\right] \tag{14}$$

$$(\Delta R)^3 = R_0^3\left[-\left(\frac{r}{R_0}\right)^3\left\{\frac{1}{4}(3\cos L + \cos 3L)\right\}\right] \tag{15}$$

Also we may write

$$\frac{\Delta R}{R_0} = -\frac{r}{R_0}\cos L\cos g + \frac{1}{8}\left(\frac{r}{R_0}\right)^2\{1-\cos 2L+\cos 2g-\cos 2L\cos 2g\}$$
$$+\frac{1}{32}\left(\frac{r}{R_0}\right)^3\{3\cos L\cos g-3\cos 3L\cos g+\cos L\cos 3g-\cos 3L\cos 3g\} \tag{16}$$

We have also

$$\frac{R_0}{R_2} = 1+\frac{r}{R_0}\cos L\cos g + \frac{1}{8}\left(\frac{r}{R_0}\right)^2\{1+3\cos 2L+\cos 2g+3\cos 2L\cos 2g\}$$
$$+\frac{1}{32}\left(\frac{r}{R_0}\right)^3\{9\cos L\cos g+15\cos 3L\cos g+3\cos L\cos 3g+5\cos 3L\cos 3g\} \tag{17}$$

where

$$R_1 = r\cos g \qquad \frac{\Delta R}{R_0} = -1+\left\{1+\left(\frac{R_1}{R_0}\right)^2-2\frac{R_1}{R_0}\cos L\right\}^{1/2} \tag{18}$$

2.1. *Investigation of the expansion for U'*

From Fig. 2, $U' = V\sin\varepsilon$ we may write

$$\frac{\partial V}{\partial z} = \frac{\partial^2 V}{\partial R\partial z} = \frac{\partial^3 V}{\partial R^2\partial z} = \frac{\partial^3 V}{\partial R\partial z^2} = 0 \tag{19}$$

$$U' = \left[V_0+\Delta R\left(\frac{\partial V}{\partial R}\right)+\frac{1}{2}(\Delta R)^2\left(\frac{\partial^2 V}{\partial R^2}\right)+\frac{1}{2}z^2\left(\frac{\partial^2 V}{\partial z^2}\right)\right.$$
$$\left.+\frac{1}{6}(\Delta R)^3\left(\frac{\partial^3 V}{\partial R^3}\right)+\frac{1}{6}z^3\left(\frac{\partial^3 V}{\partial z^3}\right)\right]\sin\varepsilon \tag{20}$$

i.e.

$$U' = \left[V_0+a\Delta R+b(\Delta R)^2+cz^2+d(\Delta R)^3+ez^3\right]\sin\varepsilon \tag{21}$$

where

$$a=\frac{\partial V}{\partial R} \quad b=\frac{1}{2}\frac{\partial^2 V}{\partial R^2} \quad c=\frac{1}{2}\frac{\partial^2 V}{\partial z^2} \quad d=\frac{1}{6}\frac{\partial^3 V}{\partial R^3} \quad e=\frac{1}{6}\left(\frac{\partial^3 V}{\partial z^3}\right) \tag{22}$$

$$\sin\varepsilon = \frac{R_1}{R_2}\sin L = \frac{r}{R_2}\cos g\sin L = \frac{R_0}{R_2}\frac{r}{R_0}\sin L\cos g \tag{23}$$

Neglecting $O(r/R_0) > 3$ we derive, after some long reductions, substitutions and arrangements, the following expression for U‘,

$$
\begin{aligned}
U' &= \frac{r}{R_0}\cos g\left\{V_0\sin L + rA\sin 2L\cos g\right\} \\
&+\left(\frac{r}{R_0}\right)^3\left[\left\{-\frac{3}{32}V_0+\frac{3}{32}aR_0+\left(\frac{3}{16}b+\frac{1}{4}c\right)R_0^2\right\}\sin L\cos g\right. \\
&+\left\{-\frac{1}{32}V_0+\frac{1}{32}aR_0+\left(\frac{1}{16}b-\frac{1}{4}c\right)R_0^2\right\}\sin L\cos 3g \\
&+\left\{\frac{9}{32}V_0-\frac{9}{32}aR_0+\frac{3}{16}bR_0^2\right\}\sin 3L\cos g \\
&\left.+\left\{\frac{3}{32}V_0-\frac{3}{32}aR_0+\frac{1}{16}bR_0^2\right\}\sin 3L\cos 3g\right]
\end{aligned} \tag{24}
$$

where

$$
A=\frac{1}{2}\left(\frac{V_0}{R_0}-a\right)\qquad a=\frac{\partial V}{\partial R}\qquad b=\frac{1}{2}\frac{\partial^2 V}{\partial R^2} \tag{25}
$$

2.2. *The Expansion of V’*

The verification runs as follows, from Fig. 2:

$$
V' = V\cos\varepsilon - V_0
$$

$$
\cos\varepsilon = 1-\frac{1}{2}\sin^2\varepsilon = 1-\frac{1}{2}\left(\frac{r}{R_0}\right)^2\sin^2 L\cos^2 g \tag{26}
$$

$$
\sin\varepsilon = \left(\frac{r}{R_0}\right)\sin L\cos g \tag{27}
$$

$$
V = V_0+\Delta R\left(\frac{\partial V}{\partial R}\right)+\frac{1}{2}(\Delta R)^2\left(\frac{\partial^2 V}{\partial R^2}\right)+\frac{1}{2}z^2\frac{\partial^2 V}{\partial z^2} \tag{28}
$$

$$
+\frac{1}{6}(\Delta R)^3\left(\frac{\partial^3 V}{\partial R^3}\right)+\frac{1}{6}z^3\frac{\partial^3 V}{\partial z^3} \tag{29}
$$

i.e.

$$
V = V_0 + a\Delta R + b(\Delta R)^2 + cz^2 + d(\Delta R)^3 + ez^3 \tag{30}
$$

Where

$$
a=\frac{\partial V}{\partial R}\qquad b=\frac{1}{2}\frac{\partial^2 V}{\partial R^2}\qquad c=\frac{1}{2}\frac{\partial^2 V}{\partial z^2}\qquad d=\frac{1}{6}\frac{\partial^3 V}{\partial R^3}\qquad e=\frac{1}{6}\frac{\partial^3 V}{\partial z^3} \tag{31}
$$

After some similar reductions and substitutions,

$$
\begin{aligned}
&V' = -ar\cos L\cos g - r^2\cos^2 g\left\{\frac{A}{R_0}\sin^2 L - b\cos^2 L - c\tan^2 g\right\}\\
&+r^3\left[\left\{\frac{3}{16}\frac{a}{R_0^2} - \frac{3}{16}\frac{b}{R_0} - \frac{9}{16}d\right\}\cos L\cos g\right.\\
&+\left\{\frac{1}{16}\frac{a}{R_0^2} - \frac{1}{16}\frac{b}{R_0} - \frac{3}{16}d\right\}\cos L\cos 3g\\
&+\left\{-\frac{3}{16}\frac{a}{R_0^2} + \frac{3}{16}\frac{b}{R_0} - \frac{3}{16}d\right\}\cos 3L\cos g\\
&\left.+\left\{-\frac{1}{16}\frac{a}{R_0^2} + \frac{1}{16}\frac{b}{R_0} - \frac{1}{16}d\right\}\cos 3L\cos 3g + e\sin^3 g\right]
\end{aligned}
\tag{32}
$$

Where $z = r\sin g$

3. Formulae for radial velocity, proper motion in galactic longitude and latitude

We have

$$\Delta\rho = U'\cos L\cos g + V'\sin L\cos g = \zeta \tag{33}$$

After some laborious substitutions, arrangements and reductions, we may write finally,

$$
\begin{aligned}
&\Delta\rho = rA\sin 2L\cos^2 g + \frac{1}{4}r^2\cos^3 g\sin L\left\{-\frac{A}{R_0} + b + 4c\tan^2 g\right\}\\
&+\frac{1}{4}r^2\cos^3 g\sin 3L\left\{3\frac{A}{R_0} + b\right\}\\
&+\left(\frac{r}{R_0}\right)^3\left[\left\{\left(\frac{3}{64}V_0 + \frac{3}{64}aR_0\right) + \frac{1}{16}cR_0^2 - \frac{3}{32}R_0^3 d\right\}\sin 2L\right.\\
&+\left\{\left(\frac{1}{16}V_0 + \frac{1}{16}aR_0\right) - \frac{1}{8}R_0^3 d\right\}\sin 2L\cos 2g\\
&+\left\{\left(\frac{1}{64}V_0 + \frac{1}{64}aR_0\right) - \frac{1}{16}cR_0^2 - \frac{1}{32}R_0^3 d\right\}\sin 2L\cos 4g+\\
&+\left\{\left(\frac{9}{128}V_0 - \frac{15}{128}aR_0\right) + \frac{3}{32}bR_0^2 - \frac{3}{64}R_0^3 d\right\}\sin 4L\\
&+\left\{\left(\frac{3}{32}V_0 - \frac{5}{32}aR_0\right) + \frac{1}{8}bR_0^2 - \frac{1}{16}R_0^3 d\right\}\sin 4L\cos 2g\\
&+\left\{\left(\frac{3}{128}V_0 - \frac{5}{128}aR_0\right) + \frac{1}{32}bR_0^2 - \frac{1}{64}R_0^3 d\right\}\sin 4L\cos 4g\\
&\left.+R_0^3 e\sin L\tan^3 g\cos^4 g\right]
\end{aligned}
\tag{34}
$$

For purposes of numerical analysis, $\Delta\rho$ can be written in the form,

$$\Delta\rho = A_1\sin L + A_2\sin 2L + A_3\sin 3L + A_4\sin 4L = \zeta$$

Where,

$$A_1 = \frac{1}{4}r^2\cos^3 g\left\{-\frac{A}{R_0}+b+4c\tan^2 g\right\}+r^3 e\tan^3 g\cos^4 g \tag{35}$$

$$\begin{aligned}A_2 &= rA\cos^2 g\\&+\left(\frac{r}{R_0}\right)^3\left[\left\{\frac{3}{64}V_0+\frac{3}{64}aR_0+\frac{1}{16}cR_0^2-\frac{3}{32}R_0^3 d\right\}\right.\\&+\left\{\frac{1}{16}V_0+\frac{1}{16}aR_0-\frac{1}{8}R_0^3 d\right\}\cos 2g\\&\left.+\left\{\frac{1}{64}V_0+\frac{1}{64}aR_0-\frac{1}{16}cR_0^2-\frac{1}{32}R_0^3 d\right\}\cos 4g\right]\end{aligned} \tag{36}$$

$$A_3 = \frac{1}{4}r^2\cos^3 g\left(3\frac{A}{R_0}+b\right) \tag{37}$$

$$\begin{aligned}A_4 &= \left(\frac{r}{R_0}\right)\left[\left\{\frac{9}{128}V_0-\frac{15}{128}aR_0+\frac{3}{32}bR_0^2-\frac{3}{64}R_0^3 d\right\}\right.\\&+\left\{\frac{3}{32}V_0-\frac{5}{32}aR_0+\frac{1}{8}bR_0^2-\frac{1}{16}R_0^3 d\right\}\cos 2g\\&\left.+\left\{\frac{3}{128}V_0-\frac{5}{128}aR_0+\frac{1}{32}bR_0^2-\frac{1}{64}R_0^3 d\right\}\cos 4g\right]\end{aligned} \tag{38}$$

3.1. ***The expression of*** $\xi = -\mathbf{U}'\sin\mathbf{L}+\mathbf{V}'\cos\mathbf{L}$

We finally reach the following formula, after tedious computations,

$$\begin{aligned}\xi &= r\cos g\left\{A\cos 2L+B+\frac{1}{4}r\cos g\cos L\left(-\frac{3A}{R_0}+3b+4e\tan^2 g\right.\right.\\&\left.\left.+4re\tan^3 g\cos g+\frac{1}{4}r\cos g\cos 3L\left(\frac{3A}{R_0}+b\right)\right)\right\}\\&+r^3\left[\left(\frac{3}{64}\frac{V_0}{R_0^3}+\frac{3}{64}\frac{a}{R_0^2}-\frac{3}{16}\frac{b}{R_0}-\frac{1}{8}\frac{c}{R_0}-\frac{9}{32}d\right)\cos g\right.\\&+\left(\frac{1}{64}\frac{V_0}{R_0^3}+\frac{1}{64}\frac{a}{R_0^2}-\frac{1}{16}\frac{b}{R_0}+\frac{1}{8}\frac{c}{R_0}-\frac{3}{32}d\right)\cos 3g\\&+\left(-\frac{3}{16}\frac{V_0}{R_0^3}+\frac{3}{16}\frac{a}{R_0^2}+\frac{1}{8}\frac{c}{R_0}-\frac{3}{8}d\right)\cos 2L\cos g\\&+\left(-\frac{1}{16}\frac{V_0}{R_0^3}+\frac{1}{16}\frac{a}{R_0^2}-\frac{1}{8}\frac{c}{R_0}-\frac{1}{8}d\right)\cos 2L\cos 3g\\&+\left(\frac{9}{64}\frac{V_0}{R_0^3}-\frac{15}{64}\frac{a}{R_0^2}+\frac{3}{16}\frac{b}{R_0}-\frac{3}{32}d\right)\cos 4L\cos g\\&\left.+\left(\frac{3}{64}\frac{V_0}{R_0^3}-\frac{5}{64}\frac{a}{R_0^2}+\frac{1}{16}\frac{b}{R_0}-\frac{1}{32}d\right)\cos 4L\cos 3g\right]\end{aligned} \tag{39}$$

where,

$$\begin{aligned} &a=\frac{\partial V}{\partial R} \qquad b=\frac{1}{2}\frac{\partial^2 V}{\partial R^2} \qquad c=\frac{1}{2}\frac{\partial^2 V}{\partial z^2} \qquad d=\frac{1}{6}\frac{\partial^3 V}{\partial R^3} \\ &e=\frac{1}{6}\frac{\partial^3 V}{\partial z^3} \qquad A=\frac{1}{2}\left(\frac{V_0}{R_0}-a\right) \qquad B=A-\frac{V_0}{R_0} \end{aligned} \tag{40}$$

3.2. *The expression of* $\eta=-\sin g\,(U'\cos L+V'\sin L)$

We may write the following formula after some hard calculations and reductions,

$$\begin{aligned} \eta = &-\frac{1}{2}rA\sin 2L\sin 2g-\frac{1}{4}r^2\sin L\sin g\cos^2 g\left(-\frac{A}{R_0}+b+4c\tan^2 g\right. \\ &\left.+4re\tan^3 g\cos g\right)-\frac{1}{4}r^2\sin 3L\sin g\cos^2 g\left(\frac{3A}{R_0}+b\right) \\ &-r^3\left[\left(\frac{1}{32}\frac{V_0}{R_0^3}+\frac{1}{32}\frac{a}{R_0^2}+\frac{1}{8}\frac{c}{R_0}-\frac{1}{16}d\right)\sin 2L\sin 2g\right. \\ &+\left(\frac{1}{64}\frac{V_0}{R_0^3}+\frac{1}{64}\frac{a}{R_0^2}-\frac{1}{16}\frac{c}{R_0}-\frac{1}{32}d\right)\sin 2L\sin 4g \\ &+\left(\frac{3}{64}\frac{V_0}{R_0^3}-\frac{5}{64}\frac{a}{R_0^2}+\frac{1}{16}\frac{b}{R_0}-\frac{1}{32}d\right)\sin 4L\sin 2g \\ &\left.+\left(\frac{3}{128}\frac{V_0}{R_0^3}-\frac{5}{128}\frac{a}{R_0^2}+\frac{1}{32}\frac{b}{R_0}-\frac{1}{64}d\right)\sin 4L\sin 4g\right] \end{aligned} \tag{41}$$

4. Discussion

Firstly, we assigned the equalities for ξ, η, ζ, the linear components of S corresponding to proper motion equations and radial velocity. We write down the equation for V up to third order terms. We calculated algebraically (ΔR), $(\Delta R)^2$, $(\Delta R)^3$, $\Delta R/R_0$, R_0/R_2 up to the order $O\,(r/R_0)^3$.

We write down the expansions for U', V' up to third power in (r/R_0).

We cited the equalities for $\zeta=\Delta\rho$ the radial velocity component, and the formulae for proper motion in galactic longitude and latitude ξ, η after some lengthy and hard calculations.

Expansions are performed up to order 3 in (r/R_0) which yield higher accuracy in numerical results and progress in the development.

We stress on the dynamical points of view of this problem in this treatment.

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