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Flow Modeling in a Porous Cylinder with Regressing Walls Using Semi Analytical Approach

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In this paper, the mathematical modeling of the flow in a porous cylinder with a focus on applications to solid rocket motors is presented. As usual, the cylindrical propellant grain of a solid rocket motor is modeled as a long tube with one end closed at the headwall, while the other remains open. The cylindrical wall is assumed to be permeable so as to simulate the propellant burning and normal gas injection. At first, the problem description and formulation are considered. The Navier–Stokes equations for the viscous flow in a porous cylinder with regressing walls are reduced to a nonlinear ODE by using a similarity transformation in time and space. Application of Differential Transformation Method (DTM) as an approximate analytical method has been successfully applied. Finally the results have been presented for various cases.

Keywords: Porous cylinder, flow modeling, solid rocket motors, analytical approach, DTM.

1. Introduction

The flow of Newtonian and non–Newtonian fluids in a porous surface channel has attracted the interest of many investigators in view of its applications in engineering practice. One of these applications is to treat the internal motion of the gases in solid rocket motors as the superposition of a steady average flow and a conglomeration of unsteady fields [1].

The average flow, also commonly known as the mean flow, represents the bulk motion of the gases in the rocket and can be approximated by the steady flow in a porous pipe. Most scientific problems such as two–dimensional viscous flow between slowly expanding or contracting walls with weak permeability and other fluid mechanic problems are inherently nonlinear. Except a limited number of these problems, most of them do not have analytical solution. Therefore, these nonlinear equations should be solved using other methods.

J. Stebel, [2] conducted a study on shape stability of incompressible fluids subject to Navier slip, focusing on the equations of motions for incompressible fluids that slip at the wall. It was noted that the issue of boundary conditions in fluid mechanics has been studied for over two centuries by many distinguished scientists but still it is subject to discussion in the mathematical community.

Makinde and Osalusi [3] investigated the steady flow in a channel with slip at the permeable boundaries. They reported that an increase in the positive value of flow Reynolds number (Re) represents an increase in the fluid suction while an increase in the negative value of Re represents an increase in the fluid injection. They also noticed that wall skin friction increases with suction and decreases with injection and that, both slip parameter and magnetic field have great influence on wall skin friction. A similar study was done by Makinde [4] on extending the utility of perturbation series in problems of laminar flow in a porous pipe and diverging channel, by considering a steady ax symmetric flow of a viscous incompressible fluid driven.

Along a pipe by the combined effect of the wall deceleration and suction. It was stated that a bifurcation occurs where the solutions of a non–linear system change their qualitative character as a parameter changes. In particular, bifurcation theory is about how the number of steady solutions of a system depends on a parameter. Yogeshi and Denn [5] conducted a study on planar contraction flow with a slip boundary condition in which they analyzed the creeping flow of Newtonian and inelastic non Newtonian fluids in a planar contraction with Navier (linear) slip boundary condition. It was found that, curved streamlines arises in the presence of wall slip, which may be a factor in the initiation of instabilities associated with entry flow.

The flow of an incompressible viscous fluid between a uniformly porous upper plate and a lower impermeable plate that is subjected to a Navier slip is modeled and analyzed in this study using analytical approaches [6–12].

2. Mathematical Formulation

Consider the laminar, isothermal and incompressible flow in a cylindrical domain bounded by permeable surfaces with one end closed at the head well while the other remains open. A schematic diagram of the problem is shown in Fig. 1. The walls expand radially at a time–dependent rate a^*. Furthermore, the origin $x^* = 0$ is assumed to be the center of the classic squeeze film problem. This enables us to assume flow symmetry about $x^* = 0$.

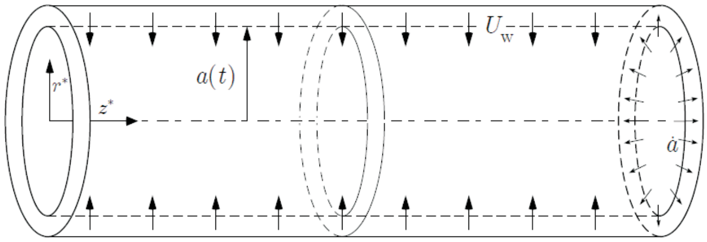

Figure 1 Schematic of problem

Under these assumptions, the transport equation for the unsteady flow is given as follows:

$$\frac{\partial \Omega^*}{\partial t} - \nabla^* \times u^* \times \Omega^* = v\nabla^* \times \nabla^{*2} u^* \tag{1}$$

$$\frac{\partial u^*}{\partial t} + u^* \frac{\partial u^*}{\partial y^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x^*} + \nu \left[\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right] \tag{2}$$

$$\frac{\partial v^*}{\partial t} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial y^*} + \nu \left[\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right] \tag{3}$$

Where p^*, ρ, v and t are the dimensional pressure, density, kinematic viscosity and time, respectively. Auxiliary conditions can be specified such as:

$$\begin{aligned} & y^* = a\,(t) : u^* = 0, \quad v^* = -V_w = -\frac{a^*}{c} \\ & y^* = 0 : \frac{\partial u^*}{\partial y^*} = 0, \quad v^* = 0 \\ & x^* = 0 : u^* = 0 \end{aligned} \tag{4}$$

Using some modification and special variable [13], and the we have:

$$F^{IV} + \alpha\,(yF''' + 3F'') + ReF'F''' - ReF'F'' = 0 \tag{5}$$

With the following boundary conditions:

$$\begin{aligned} y = 0 : f = 0, \quad f' = 0 \\ y = 1 : f = 1, \quad f' = 0 \end{aligned} \tag{6}$$

The resulting Eqn. 5 is the classic Berman's formula [14], with $\alpha = 0$ (channel with stationary walls). After the flow field is found, the normal pressure gradient can be obtained by substituting the velocity components into Eqns 1–3. Hence it is:

$$p_y = -\left[Re^{-1} f'' + ff' + \alpha Re^{-1}\,(f + yf')\right], \quad p = \frac{p^*}{\rho V_w^2} \tag{7}$$

Introducing the non–dimensional shear stress $\tau = \frac{\tau^*}{\rho V_w^2}$, we have:

$$\tau = \frac{xf''}{Re} \tag{8}$$

3. Solution Procedure

Let $x(t)$ be analytic in a domain D and let $t = t_i$ represents any point in D. The function $x(t)$ is then represented by one power series whose center is located at t_i. The Taylor series expansion function of $x(t)$ is in the form of:

$$x(t) = \sum_{k=0}^{\infty} \frac{(t-t_i)}{k!} \left[\frac{d^k x(t)}{dt^k}\right]_{t=t_i} \quad \forall t \in D \tag{9}$$

As explained in [4] the differential transformation of the function $x(t)$ is defined as follows:

$$X(k) = \sum_{k=0}^{\infty} \frac{H^k}{k!} \left[\frac{d^k x(t)}{dt^k}\right]_{t=0} \tag{10}$$

Where $x(t)$ is the original function and $X(k)$ is the transformed function. The differential inverse transform of $X(k)$is defined as follows:

$$x(t) = \sum_{k=0}^{\infty} \left(\frac{t}{H!}\right)^k X(k) \tag{11}$$

Mathematical operations performed by DTM are listed in Tab. 1.

Table 1 Some of the basic operations of DTM

Original function	Transformed function
$x(t) = \alpha f(t) \pm \beta g(t)$	$X(k) = \alpha F(k) \pm \beta G(k)$
$x(t) = f(t)\, g(t)$	$X(k) = \sum_{l=0}^{k} F(l)\, G(k-l)$
$x(t) = \frac{df(t)}{dt}$	$X(k) = (k+1)\, F(k+1)$
$x(t) = \frac{d^2 f(t)}{dt^2}$	$X(k) = (k+1)(k+2)\, F(k+2)$

Taking differential transform of Eqn. 10 by using the related definitions in Tab. 1, we obtain:

$$\begin{aligned}
&(k+1)(k+2)(k+3)(k+4)\, U(k+4) \\
&+\alpha \left(\sum_{l=0}^{k} (k-l+1)(k-l+2)(k-l+3)\, U(k-l+3)\, \delta(l-1)\right) \\
&+3\alpha (k+1)(k+2)\, U(k+2) \\
&+Re \sum_{l=0}^{k} (k-l+1)(k-l+2)(k-l+3)\, U(k-l+3)\, U(l) \\
&-Re \sum_{l=0}^{k} (l+1)(k-l+1)(k-l+2)\, U(k-l+2)\, U(l+1) = 0
\end{aligned} \tag{12}$$

In order to solve Eqn. 12, we consider the following boundary conditions:

$$U(0) = 0, \quad U(2) = 0, \quad \sum_{l=0}^{k} U(l) = 1, \quad \sum_{l=0}^{k} lU(l) = 0 \tag{13}$$

However it can be yield that the closed form of the solutions is:

$$U(t) = U(0) \times t^0 + U(1) \times t^1 + U(2) \times t^2 + \cdots \tag{14}$$

4. Results and Discussions

The objective of the present study was to apply DTM to obtain an explicit analytic solution of laminar, isothermal, incompressible viscous flow in a rectangular domain bounded by two moving porous walls, which enable the fluid to enter or exit.

Fig. 2 shows the effects of changing the Reynolds number while maintaining the values of Non-dimensional wall dilation rate. The result shows that as the Reynolds number increases, the normal component of velocity decrease. In Fig. 2 a proper comparison is also made between the numerical solution obtained by Runge Kutta method and RVIM. A great agreement between analytical solutions and numerical ones are illustrated.

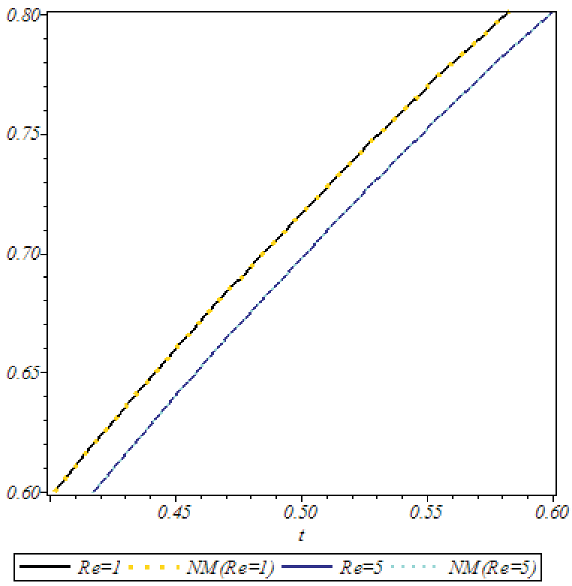

Figure 2 Effect of Reynolds number on axial velocity

In Fig. 3, the effects of non–dimensional wall dilation rate with constant Reynolds number on radial velocity can be illustrated.

For every level of injection or suction, in the case of expanding wall, increasing α leads to higher radial velocity near the center and the lower radial velocity near the wall. The reason is that the flow toward the center becomes greater to make up for the space caused by the expansion of the wall and as a result, the radial velocity also becomes greater near the center

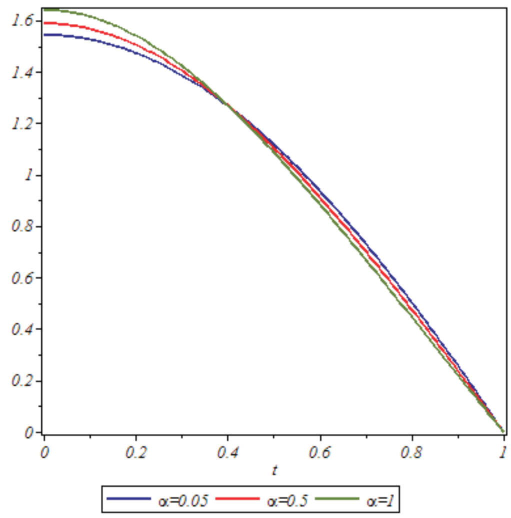

Figure 3 Effect of wall dilation rate number on radial velocity

In Fig. 4 the pressure drop for case $Re = 10$ can be illustrated. The effects of dilation number is also seen through plot.

Fig. 4 shows that for every level of injection or suction, the absolute pressure change in the normal direction is lowest near the central portion. Furthermore, by increasing non–dimensional wall dilation rates the absolute value of pressure distribution in the normal direction increases.

Non–dimensional wall dilation rates, are plotted in Fig. 5. We can observe from Fig. 5 that the absolute shear stress along the wall surface increases in proportion to x. Furthermore, by increasing nondimensional wall dilation rates the absolute value of shear stress increases.

5. Conclusions

A study of an incompressible two–dimensional flow in a channel with one porous wall is presented in this research. The governing continuity and momentum equations together with the associated boundary conditions are first reduced to a set of self similar non–linear coupled ordinary differential equations using similarity transformations. Then we solved the ordinary differential equation by DTM and the numerical method.

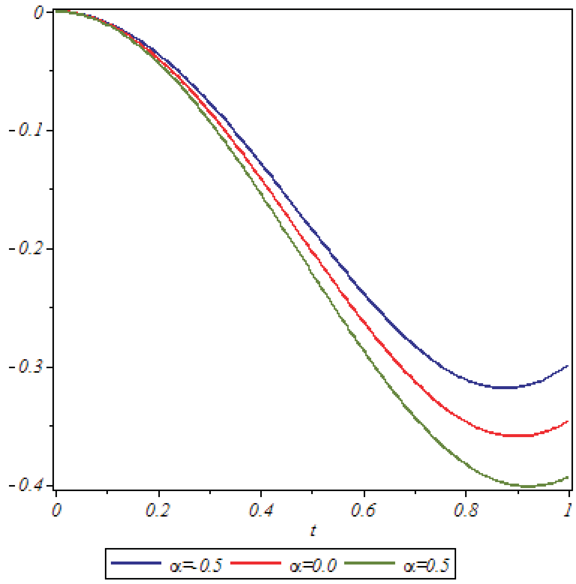

Figure 4 Pressure drop for various values of dilation number

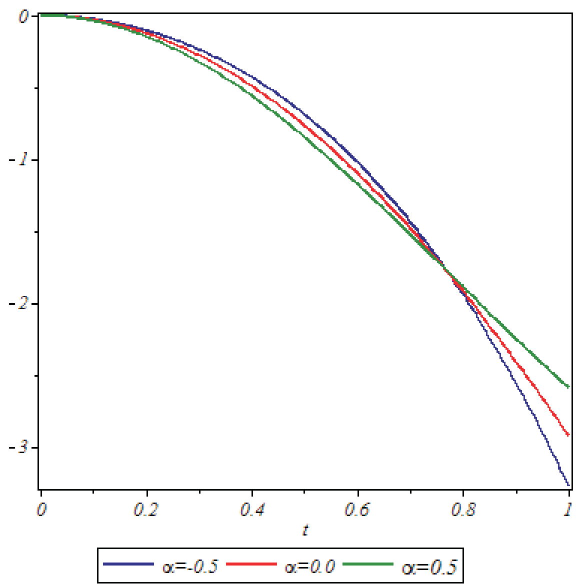

Figure 5 Shear stress changes shown over a range of dilation rate

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