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A Semi – Analytic First Order Jupiter – Saturn Planetary Theory Part I : Outline

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Herein we lay the broad lines for the construction of a first order w.r.t planetary masses Jupiter–Saturn theory – giving the orbital elements of the two planets at any epoch. This is implemented by the evaluation of the R. H. S. of the original first order Hamiltonian equations of motion. The first order Hamiltonian is composed of the first order secular terms and the first order periodic terms. We restrict the periodic terms to be the commensurate ones for the J–S (Jupiter–Saturn) subsystem. We give throughout the text an idea to the extension of the theory to the case of the four major outer planets J–S–U–N.

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1. Equations of Motion

The original first order Hamiltonian equations of motion, in terms of the variables of H. Poincare' for the J–S subsystem is given by

$$\begin{aligned}
\frac{dL_u}{dt} &= \frac{\partial F_1}{\partial \lambda_u} \qquad & \frac{d\lambda_u}{dt} &= -\frac{\partial F_1}{\partial L_u} \\
\frac{dH_u}{dt} &= \frac{\partial F_1}{\partial K_u} \qquad & \frac{dK_u}{dt} &= -\frac{\partial F_1}{\partial H_u} \\
\frac{dP_u}{dt} &= \frac{\partial F_1}{\partial Q_u} \qquad & \frac{dQ_u}{dt} &= -\frac{\partial F_1}{\partial P_u}
\end{aligned} \tag{1}$$

Where L, λ, H, K, P, Q are the Poincare‘ variables given by:

$$
\begin{aligned}
L_u &= beta_u \sqrt{k^2 m_0 m_{0u} a_u} \\
H_u &= \sqrt{2L_u\left(1-\sqrt{1-e_u^2}\right)}\cos\varpi_u \\
P_u &= \sqrt{2L_u\sqrt{1-e_u^2}\,(1-\cos i_u)}\cos\Omega_u \\
\lambda_u &= l_u + \varpi_u \\
K_u &= -\sqrt{2L_u\left(1-\sqrt{1-e_u^2}\right)}\sin\varpi_u \\
Q_u &= -\sqrt{2L_u\sqrt{1-e_u^2}\,(1-\cos i_u)}\sin\Omega_u
\end{aligned}
\tag{2}
$$

F_1 – the first order generating Hamiltonian with respect to planetary masses,

$$F_1 = F_{1S} + F_{1P} \tag{3}$$

F_{1S} is the first order secular Hamiltonian; F_{1P} is the first order periodic Hamiltonian restricted to the small divisor terms, $u = 5, 6$ whereas 5 refers to Jupiter and 6 refers to Saturn.

So, we may cite:

$$
\begin{aligned}
\frac{d(L_u,H_u,P_u)}{dt} &= \frac{\partial(F_{1S}+F_{1P})}{\partial(\lambda_u,K_u,Q_u)} \\
\frac{d(\lambda_u,K_u,Q_u)}{dt} &= -\frac{\partial(F_{1S}+F_{1P})}{\partial(L_u,H_u,P_u)}
\end{aligned}
\tag{4}
$$

2. Method and results

In general

$$F_{1S} = \sum_{j=1}^{n} \frac{k^4 m_0^2 m_{0j}\beta_j^3}{2L_j^2} + \text{secular part of } \sigma \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{k^4 m_0 m_{0j}\beta_i\beta_j^3}{L_j^2}\left(\frac{a_j}{\Delta_{ij}}\right) \tag{5}$$

with

$$a_j = \frac{L_j^2}{k^2 m_0 m_{0j}\beta_j^2}$$

where

k – The Gaussian constant
m_0 – mass of the Sun
a_j – semi – major axis of planet j
σ – small parameter of the order of planetary masses,
taken equal to 10^{-3}
$\sigma \quad \beta_j = m_j$ – mass of planet j
$m_{0j} = (m_0 + m_1 + \ldots + m_{j-1}) \;/\; (m_0 + m_1 + \ldots + m_j)$
Δ_{ij} – mutual distance between planet i,j

Thus for the subsystem J–S, we write

$$
\begin{aligned}
F_{1S} &= \psi_0 \left(L_u,\, H_u,\, K_u,\, P_u,\, Q_u \right) = \frac{const}{L_5^2} + \frac{const}{L_6^2} \\
&+ \frac{const}{L_6^2} \left\{ \begin{array}{l} V_1 + V_2 \left(H_5^2 + K_5^2 \right) + V_3 \left(P_5^2 + Q_5^2 \right) + \\ V_4 \left(H_6^2 + K_6^2 \right) + V_5 \left(P_6^2 + Q_6^2 \right) + \\ V_6 \left(P_5 P_6 + Q_5 Q_6 \right) + V_7 \left(H_5 H_6 + K_5 K_6 \right) \end{array} \right\}
\end{aligned} \tag{6}
$$

Where the V's are functions of $L_5,\, L_6$; or in an other explicit form, we may write for the J–S subsystem,

$$
F_{1S} = \psi_0 \left(L_u,\, H_u,\, K_u,\, P_u,\, Q_u \right) = \frac{k^4 m_0^2 m_{05} \beta_5^3}{2L_5^2} + \frac{k^4 m_0^2 m_{06} \beta_6^3}{2L_6^2} \tag{7}
$$

$$
+ \frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[\begin{array}{l} \frac{1}{2} f_1^{(5,6)} + \frac{1}{8L_5} f_2^{(5,6)} \left(H_5^2 + K_5^2 \right) - \frac{1}{8L_5} f_3^{(5,6)} \left(P_5^2 + Q_5^2 \right) + \\ \frac{1}{8L_6} f_2^{(5,6)} \left(H_6^2 + K_6^2 \right) - \frac{1}{8L_6} f_3^{(5,6)} \left(P_6^2 + Q_6^2 \right) + \\ \frac{1}{4\sqrt{L_5 L_6}} f_3^{(5,6)} \left(P_5 P_6 + Q_5 Q_6 \right) + \frac{1}{4\sqrt{L_5 L_6}} f_9^{(5,6)} \left(H_5 H_6 + K_5 K_6 \right) \end{array} \right]
$$

Neglecting powers > 2 in the Poincare‘ variables.

The f‘s are functions in $L_5,\, L_6$.

Whence the constants and the V's in eq.(6) can be easily identified.

Generally for the case of the four outer major planets (J–S–U–N), Jupiter, Saturn, Uranus and Neptune, we may write

$$
F_{1P} = \delta_1 F_1 + \delta_2 F_1 + \delta_3 F_1 \tag{8}
$$

Where

$$
\delta_1 F_1 = A \left[\cos \left(5\lambda_6 - 2\lambda_5 \right) \psi_1 + \sin \left(5\lambda_6 - 2\lambda_5 \right) \psi_2 \right] \tag{9}
$$

$$
\delta_2 F_1 = B \left[\cos \left(2\lambda_8 - \lambda_7 \right) \psi_3 + \sin \left(2\lambda_8 - \lambda_7 \right) \psi_4 \right] \tag{10}
$$

$$
\delta_3 F_1 = C \left[\cos \left(2\lambda_8 - \lambda_7 \right) \psi_5 + \sin \left(2\lambda_8 - \lambda_7 \right) \psi_6 \right] \tag{11}
$$

Numbers 7, 8 refer to Uranus and Neptune respectively.

In the case of the construction of a (J–S) planetary theory, we are capable of neglecting $\delta_2 F_1$, $\delta_3 F_1$, since subscripts 5, 6 are totally absent in all terms of the two equations (10, 11). A, B, C are given by the following equalities:

$$
A = \frac{\sigma k^2 \beta_5 \beta_6}{a_6} \qquad B = \frac{\sigma k^2 \beta_7 \beta_8}{a_8} \qquad C = \frac{\sigma k^2 m_0 \beta_7 \beta_8 a_7}{a_8^2} \tag{12}
$$

Whence from the above we should write for the J–S subsystem:

$$
F_{1P} = \delta_1 F_1 = A \left[\cos \left(5\lambda_6 - 2\lambda_5 \right) \psi_1 + \sin \left(5\lambda_6 - 2\lambda_5 \right) \psi_2 \right] \tag{13}
$$

ψ_1 , ψ_2 are polynomial functions of the P.V.; and are given by

$$\psi_1 = \psi_1 (L_u, H_u, K_u, P_u, Q_u) = \left[\begin{array}{l} U_1 \left(H_5^3 - 3H_5K_5^2\right) + U_2 \left(H_6H_5^2 - H_6K_5^2 - 2H_5K_5K_6\right) + \\ U_3 \left(H_5H_6^2 - H_5K_6^2 - 2H_6K_5K_6\right) + U_4 \left(H_6^3 - 3H_6K_6^2\right) + \\ U_5 \left(H_5P_5^2 - H_5Q_5^2 - 2K_5P_5Q_5\right) + \\ U_6 \left(H_5P_6^2 - H_5Q_6^2 - 2K_5P_6Q_6\right) + \\ U_7 (K_5P_6Q_5 + K_5P_5Q_6 - H_5P_5P_6 + H_5Q_5Q_6) + \\ U_8 \left(H_6P_5^2 - H_6Q_5^2 - 2K_6P_5Q_5\right) + \\ U_9 \left(H_6P_6^2 - H_6Q_6^2 - 2K_6P_6Q_6\right) + \\ U_{10} (K_6P_6Q_5 + K_6P_5Q_6 - H_6P_5P_6 + H_6Q_5Q_6) \\ \end{array}\right] \tag{14}$$

$$\psi_2 = \psi_2 (L_u, H_u, K_u, P_u, Q_u) = \left[\begin{array}{l} U_{11} \left(K_5^3 - 3K_5H_5^2\right) + U_{12} \left(-K_6H_5^2 + K_6K_5^2 - 2H_5H_6K_5\right) + \\ U_{13} \left(-K_5H_6^2 + K_5K_6^2 - 2H_5H_6K_6\right) + U_{14} \left(K_6^3 - 3K_6H_6^2\right) + \\ U_{15} \left(-K_5P_5^2 + K_5Q_5^2 - 2H_5P_5Q_5\right) + \\ U_{16} \left(K_5Q_6^2 - K_5P_6^2 - 2H_5P_6Q_6\right) + \\ U_{17} (K_5P_5P_6 - K_5Q_5Q_6 + H_5P_6Q_5 + H_5P_5Q_6) + \\ U_{18} \left(K_6Q_5^2 - K_6P_5^2 - 2H_6P_5Q_5\right) + \\ U_{19} \left(K_6Q_6^2 - K_6P_6^2 - 2H_6P_6Q_6\right) + \\ U_{20} (K_6P_5P_6 - K_6Q_5Q_6 + H_6Q_5P_6 + H_6P_5Q_6) \\ \end{array}\right] \tag{15}$$

$U_1, U_2, ..., U_{20}$ are functions of L_5, L_6.

The final required result may be obtained from the simultaneous solution of the following 12 equations of motion, analytically or numerically or both,

$$\begin{aligned} \frac{dL_u}{dt} &= \frac{\partial\psi_0}{\partial\lambda_u} + \frac{\partial}{\partial\lambda_u} \left[\{\psi_1 \cos(5\lambda_6 - 2\lambda_5) + \psi_2 \sin(5\lambda_6 - 2\lambda_5)\}\right] \\ \frac{d\lambda_u}{dt} &= -\frac{\partial\psi_0}{\partial L_u} - \frac{\partial\psi_1}{\partial L_u}\cos(5\lambda_6 - 2\lambda_5) - \frac{\partial\psi_2}{\partial L_u}\sin(5\lambda_6 - 2\lambda_5) \\ \frac{dH_u}{dt} &= \frac{\partial\psi_0}{\partial K_u} + \frac{\partial\psi_1}{\partial K_u}\cos(5\lambda_6 - 2\lambda_5) + \frac{\partial\psi_2}{\partial K_u}\sin(5\lambda_6 - 2\lambda_5) \\ \frac{dK_u}{dt} &= -\frac{\partial\psi_0}{\partial H_u} - \frac{\partial\psi_1}{\partial H_u}\cos(5\lambda_6 - 2\lambda_5) - \frac{\partial\psi_2}{\partial H_u}\sin(5\lambda_6 - 2\lambda_5) \\ \frac{dP_u}{dt} &= \frac{\partial\psi_0}{\partial Q_u} + \frac{\partial\psi_1}{\partial Q_u}\cos(5\lambda_6 - 2\lambda_5) + \frac{\partial\psi_2}{\partial Q_u}\sin(5\lambda_6 - 2\lambda_5) \\ \frac{dQ_u}{dt} &= -\frac{\partial\psi_0}{\partial P_u} - \frac{\partial\psi_1}{\partial P_u}\cos(5\lambda_6 - 2\lambda_5) - \frac{\partial\psi_2}{\partial P_u}\sin(5\lambda_6 - 2\lambda_5) \\ u &= 5, 6 \end{aligned} \tag{16}$$

The simultaneous solution of the above 12 equations, Eqs (16), renders the values of $L = L(t), \lambda = \lambda(t), H = H(t), K = K(t), P = P(t), Q = Q(t)$, where t is the physical time. The orbital elements are derived easily from the following equalities:

$$
\begin{aligned}
a_i &= \frac{L_i^2}{\beta_i^2 m_{0i} k^2 m_0} \\
m_{0i} &= \frac{m_0 + m_1 + \ldots + m_{i-1}}{m_0 + m_1 + \ldots + m_{i-1} + m_i} \\
e_i &= \sqrt{1 - \left(1 - \frac{H_i^2 + K_i^2}{2L_i}\right)^2} \\
i_i &= \cos^{-1}\left[1 - \frac{P_i^2 + Q_i^2}{2L_i\left(1 - \frac{H_i^2+K_i^2}{2L_i}\right)}\right] \\
\varpi_i &= \tan^{-1}\left(-\frac{K_i}{H_i}\right) \\
\Omega_i &= \tan^{-1}\left(-\frac{Q_i}{P_i}\right) \\
\varepsilon_i &= \lambda_i - n_i t
\end{aligned}
$$

3. Discussion

In the future we shall give much more algebraic detailed computations, concerned with the J–S theory, and the method for the simultaneous solution of the final equations (16) either analytically or numerically or both. Obviously no indication to canonical transformations or Von Zeipel– Hori–Lie techniques, to solve the equations of motion analytically, is given. The author investigated these two techniques in detail in previous published research papers.

The present approach reduces the analytic calculations and the complexity of the problem to a great extent an advantage.

The small divisor terms are assigned from Brouwer–Clemence expansions of the principal and indirect parts of the perturbing function.

Brouwer's expansions are in terms of the mutual inclination of any two planets, which is a restricted special case. We prefer the reference of the inclinations, in the general case of more than two planets, to be to a common fixed plane. This requires using the transformation formulae of Jacobi–Radau set of origins.

We should use the expansion of Δ^{-s}, s =1, 3, 5, 7, ... when proceeding to a higher order (J–S) theory (See ref. 8 in the Bibl.).

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