A Semi–Analytic First Order
Jupiter – Saturn Planetary Theory
Part II

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In this part we present the algebraic calculus computations related to the Hamiltonian equations of motion for the Jupiter – Saturn subsystem. Also we give a comment on the methods of the solution for the system of linear and nonlinear differential equations describing the motion of this subsystem.

Keywords: Dynamics of the Solar System, Jupiter – Saturn planetary theory.

1. Methods and Results

1.1. *Equations of Motion*

For the J – S subsystem, the original first order Hamiltonian equations of motion, in terms of the H. Poincare' variables are given by:

$$\frac{dL_u}{dt} = \frac{\partial F_1}{\partial \lambda_u} \qquad \frac{d\lambda_u}{dt} = -\frac{\partial F_1}{\partial L_u}$$

$$\frac{dH_u}{dt} = \frac{\partial F_1}{\partial K_u} \qquad \frac{dK_u}{dt} = -\frac{\partial F_1}{\partial H_u} \tag{1}$$

$$\frac{dP_u}{dt} = \frac{\partial F_1}{\partial Q_u} \qquad \frac{dQ_u}{dt} = -\frac{\partial F_1}{\partial P_u}$$

$L, \lambda, H, K, P, Q$are the Poincare' variables. $u = 5$, 6 for Jupiter and Saturn.

$$L_u = \beta_u \sqrt{k^2 m_0 m_{0u} a_u}$$

$$\lambda_u = l_u + \varpi_u$$

$$H_u = \sqrt{2L_u \left(1 - \sqrt{1 - e_u^2}\right)} \cos \varpi_u$$

$$K_u = -\sqrt{2L_u \left(1 - \sqrt{1 - e_u^2}\right)} \sin \varpi_u \qquad (2)$$

$$P_u = \sqrt{2L_u \sqrt{1 - e_u^2} \left(1 - \cos i_u\right)} \cos \Omega_u$$

$$Q_u = -\sqrt{2L_u \sqrt{1 - e_u^2} \left(1 - \cos i_u\right)} \sin \Omega_u$$

$$F_1 = F_{1s} + F_{1p} \qquad (3)$$

where:

F_{1s} – first order secular Hamiltonian,

F_{1p} – first order periodic Hamiltonian.

So we may cite:

$$\frac{d\left(L_u, H_u, P_u\right)}{dt} = \frac{\partial \left(F_{1s} + F_{1p}\right)}{\partial \left(\lambda_u, K_u, Q_u\right)}$$

$$\frac{d\left(\lambda_u, K_u, Q_u\right)}{dt} = -\frac{\partial \left(F_{1s} + F_{1p}\right)}{\partial \left(L_u, H_u, P_u\right)} \qquad (u = 5, 6)$$

$$(4)$$

1.2. *Algebraic Calculus Computations*

We have:

$$\frac{dH_u}{dt} = \frac{\partial \psi_0}{\partial K_u} + \frac{\partial \psi_1}{\partial K_u} \cos\left(5\lambda_6 - 2\lambda_5\right) + \frac{\partial \psi_2}{\partial K_u} \sin\left(5\lambda_6 - 2\lambda_5\right) \tag{5}$$

$$\frac{\partial \psi_0}{\partial K_5} = \frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[\frac{1}{4L_5} f_2^{(5,6)} K_5 + \frac{1}{4\sqrt{L_5 L_6}} f_9^{(5,6)} K_6\right] \tag{6}$$

$$\frac{\partial \psi_0}{\partial K_6} = \frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[\frac{1}{4L_6} f_2^{(5,6)} K_6 + \frac{1}{4\sqrt{L_5 L_6}} f_9^{(5,6)} K_5\right] \tag{7}$$

$$\frac{\partial \psi_1}{\partial K_5} = -U_1\left(6H_5 K_5\right) - U_2\left(2H_6 K_5 + 2H_5 K_6\right) - U_3\left(2H_6 K_6\right)$$

$$-U_5\left(2P_5 Q_5\right) - U_6\left(2P_6 Q_6\right) + U_7\left(P_5 Q_6 + P_6 Q_5\right) \tag{8}$$

$$\frac{\partial \psi_1}{\partial K_6} = -U_2\left(2H_5 K_5\right) - U_3\left(2H_6 K_5 + 2H_5 K_6\right) - U_4\left(6H_6 K_6\right)$$

$$-U_8\left(2P_5 Q_5\right) - U_9\left(2P_6 Q_6\right) + U_{10}\left(P_5 Q_6 + P_6 Q_5\right) \tag{9}$$

$$\frac{\partial \psi_2}{\partial K_5} = U_{11}\left(3K_5^2 - 3H_5^2\right) + U_{12}\left(2K_5 K_6 - 2H_5 H_6\right) + U_{13}\left(-H_6^2 + K_6^2\right)$$

$$+U_{15}\left(-P_5^2 + Q_5^2\right) + U_{16}\left(-P_6^2 + Q_6^2\right) + U_{17}\left(P_5 P_6 - Q_5 Q_6\right) \tag{10}$$

$$\frac{\partial \psi_2}{\partial K_6} = U_{12}\left(-H_5^2 + K_5^2\right) + U_{13}\left(-2H_5 H_6 + 2K_5 K_6\right) + U_{14}\left(-3H_6^2 + 3K_6^2\right)$$

$$+U_{18}\left(-P_5^2 + Q_5^2\right) + U_{19}\left(-P_6^2 + Q_6^2\right) + U_{20}\left(P_5 P_6 - Q_5 Q_6\right) \tag{11}$$

whence:

$$\frac{dH_5}{dt} = \frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[\frac{1}{4L_5} f_2^{(5,6)} K_5 + \frac{1}{4\sqrt{L_5 L_6}} f_9^{(5,6)} K_6\right]$$

$$+ \left\{-U_1\left(6H_5 K_5\right) - U_2\left(2H_6 K_5 + 2H_5 K_6\right) - U_3\left(2H_6 K_6\right) - U_5\left(2P_5 Q_5\right)\right.$$

$$\left.-U_6\left(2P_6 Q_6\right) + U_7\left(P_5 Q_6 + P_6 Q_5\right)\right\} \cos(5\lambda_6 - 2\lambda_5)$$

$$+ \left\{U_{11}\left(3K_5^2 - 3H_5^2\right) + U_{12}\left(2K_5 K_6 - 2H_5 H_6\right) + U_{13}\left(-H_6^2 + K_6^2\right)\right.$$

$$+U_{15}\left(-P_5^2 + Q_5^2\right) + U_{16}\left(-P_6^2 + Q_6^2\right)$$

$$\left.+U_{17}\left(P_5 P_6 - Q_5 Q_6\right)\right\} \sin(5\lambda_6 - 2\lambda_5) \tag{12}$$

and

$$\frac{dH_6}{dt} = \frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[\frac{1}{4L_6} f_2^{(5,6)} K_6 + \frac{1}{4\sqrt{L_5 L_6}} f_9^{(5,6)} K_5 \right]$$

$$+ \{ -U_2 \left(2H_5 K_5 \right) - U_3 \left(2H_6 K_5 + 2H_5 K_6 \right) - U_4 \left(6H_6 K_6 \right)$$

$$- U_8 \left(2P_5 Q_5 \right) - U_9 \left(2P_6 Q_6 \right) + U_{10} \left(P_5 Q_6 + P_6 Q_5 \right) \} \cos(5\lambda_6 - 2\lambda_5)$$

$$\{ +U_{12} \left(-H_5^2 + K_5^2 \right) + U_{13} \left(-2H_5 H_6 + 2K_5 K_6 \right) \tag{13}$$

$$+ U_{14} \left(-3H_6^2 + 3K_6^2 \right) + U_{18} \left(-P_5^2 + Q_5^2 \right) + U_{19} \left(-P_6^2 + Q_6^2 \right)$$

$$+ U_{20} \left(P_5 P_6 - Q_5 Q_6 \right) \} \sin(5\lambda_6 - 2\lambda_5)$$

Neglecting terms of the third degree in *H, K, P, Q.*
Similarly,

$$\frac{dK_u}{dt} = -\frac{\partial \psi_0}{\partial H_u} - \frac{\partial \psi_1}{\partial H_u} \cos \left(5\lambda_6 - 2\lambda_5 \right) - \frac{\partial \psi_2}{\partial H_u} \sin \left(5\lambda_6 - 2\lambda_5 \right) \tag{14}$$

$$-\frac{\partial \psi_0}{\partial H_5} = -\frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[\frac{1}{4L_5} f_2^{(5,6)} H_5 + \frac{1}{4\sqrt{L_5 L_6}} f_9^{(5,6)} H_6 \right] \tag{15}$$

$$-\frac{\partial \psi_0}{\partial H_6} = -\frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[\frac{1}{4L_6} f_2^{(5,6)} H_6 + \frac{1}{4\sqrt{L_5 L_6}} f_9^{(5,6)} H_5 \right] \tag{16}$$

$$-\frac{\partial \psi_1}{\partial H_5} = U_1 \left(3K_5^2 - 3H_5^2 \right) + U_2 \left(2K_5 K_6 - 2H_5 H_6 \right) + U_3 \left(-H_6^2 + K_6^2 \right)$$

$$+ U_5 \left(-P_5^2 + Q_5^2 \right) + U_6 \left(-P_6^2 + Q_6^2 \right) + U_7 \left(P_5 P_6 - Q_5 Q_6 \right) \tag{17}$$

$$-\frac{\partial \psi_1}{\partial H_6} = U_2 \left(K_5^2 - H_5^2 \right) + U_3 \left(2K_5 K_6 - 2H_5 H_6 \right) + U_4 \left(-3H_6^2 + 3K_6^2 \right)$$

$$+ U_8 \left(-P_5^2 + Q_5^2 \right) + U_9 \left(-P_6^2 + Q_6^2 \right) + U_{10} \left(P_5 P_6 - Q_5 Q_6 \right) \tag{18}$$

$$-\frac{\partial \psi_2}{\partial H_5} = U_{11} \left(6H_5 K_5 \right) + U_{12} \left(2H_6 K_5 + 2H_5 K_6 \right) + U_{13} \left(2H_6 K_6 \right) + U_{15} \left(2P_5 Q_5 \right)$$

$$+ U_{16} \left(2P_6 Q_6 \right) - U_{17} \left(P_5 Q_6 + P_6 Q_5 \right) \tag{19}$$

$$-\frac{\partial \psi_2}{\partial H_6} = U_{12} \left(2H_5 K_5 \right) + U_{13} \left(2H_6 K_5 + 2H_5 K_6 \right) + U_{14} \left(6H_6 K_6 \right) + U_{18} \left(2P_5 Q_5 \right)$$

$$+ U_{19} \left(2P_6 Q_6 \right) - U_{20} \left(P_5 Q_6 + P_6 Q_5 \right) \tag{20}$$

Whence:

$$\frac{dK_5}{dt} = -\frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[\frac{1}{4L_5} f_2^{(5,6)} H_5 + \frac{1}{4\sqrt{L_5 L_6}} f_9^{(5,6)} H_6 \right]$$

$$+ \left\{ U_1 \left(3K_5^2 - 3H_5^2 \right) + U_2 \left(2K_5 K_6 - 2H_5 H_6 \right) \right.$$

$$+U_3 \left(-H_6^2 + K_6^2 \right) + U_5 \left(-P_5^2 + Q_5^2 \right) + U_6 \left(-P_6^2 + Q_6^2 \right)$$

$$\left. +U_7 \left(P_5 P_6 - Q_5 Q_6 \right) \right\} \cos\left(5\lambda_6 - 2\lambda_5 \right)$$

$$+ \left\{ U_{11} \left(6H_5 K_5 \right) + U_{12} \left(2H_6 K_5 + 2H_5 K_6 \right) + U_{13} \left(2H_6 K_6 \right) \right. \quad (21)$$

$$+U_{15} \left(2P_5 Q_5 \right) + U_{16} \left(2P_6 Q_6 \right)$$

$$\left. -U_{17} \left(P_5 Q_6 + P_6 Q_5 \right) \right\} \sin\left(5\lambda_6 - 2\lambda_5 \right)$$

$$\frac{dK_6}{dt} = -\frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[\frac{1}{4L_6} f_2^{(5,6)} H_6 + \frac{1}{4\sqrt{L_5 L_6}} f_9^{(5,6)} H_5 \right]$$

$$+ \left\{ U_2 \left(-H_5^2 + K_5^2 \right) + U_3 \left(-2H_5 H_6 + 2K_5 K_6 \right) \right.$$

$$+U_4 \left(-3H_6^2 + 3K_6^2 \right) + U_8 \left(-P_5^2 + Q_5^2 \right) + U_9 \left(-P_6^2 + Q_6^2 \right)$$

$$\left. +U_{10} \left(P_5 P_6 - Q_5 Q_6 \right) \right\} \cos(5\lambda_6 - 2\lambda_5)$$

$$+ \left\{ U_{12} \left(2H_5 K_5 \right) + U_{13} \left(2H_6 K_5 + 2H_5 K_6 \right) + U_{14} \left(6H_6 K_6 \right) \right. \quad (22)$$

$$+U_{18} \left(2P_5 Q_5 \right) + U_{19} \left(2P_6 Q_6 \right)$$

$$\left. -U_{20} \left(P_5 Q_6 + P_6 Q_5 \right) \right\} \sin(5\lambda_6 - 2\lambda_5)$$

We have also

$$\frac{dP_u}{dt} = \frac{\partial \psi_0}{\partial Q_u} + \frac{\partial \psi_1}{\partial Q_u} \cos\left(5\lambda_6 - 2\lambda_5 \right) + \frac{\partial \psi_2}{\partial Q_u} \sin\left(5\lambda_6 - 2\lambda_5 \right) \quad (23)$$

$$\frac{\partial \psi_0}{\partial Q_5} = \frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[-\frac{1}{4L_5} f_3^{(5,6)} Q_5 + \frac{1}{4\sqrt{L_5 L_6}} f_3^{(5,6)} Q_6 \right] \quad (24)$$

$$\frac{\partial \psi_0}{\partial Q_6} = \frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[-\frac{1}{4L_6} f_3^{(5,6)} Q_6 + \frac{1}{4\sqrt{L_5 L_6}} f_3^{(5,6)} Q_5 \right] \quad (25)$$

$$\frac{\partial \psi_1}{\partial Q_5} = -U_5 \left(2H_5 Q_5 + 2K_5 P_5 \right) + U_7 \left(K_5 P_6 + H_5 Q_6 \right)$$
$$\qquad (26)$$
$$-U_8 \left(2H_6 Q_5 + 2K_6 P_5 \right) + U_{10} \left(K_6 P_6 + H_6 Q_6 \right)$$

$$\frac{\partial \psi_1}{\partial Q_6} = -U_6 \left(2H_5 Q_6 + 2K_5 P_6 \right) + U_7 \left(K_5 P_5 + H_5 Q_5 \right)$$
$$\qquad (27)$$
$$-U_9 \left(2H_6 Q_6 + 2K_6 P_6 \right) + U_{10} \left(K_6 P_5 + H_6 Q_5 \right)$$

$$\frac{\partial \psi_2}{\partial Q_5} = U_{15} \left(-2H_5 P_5 + 2K_5 Q_5 \right) + U_{17} \left(-K_5 Q_6 + H_5 P_6 \right)$$
$$\qquad (28)$$
$$+U_{18} \left(-2H_6 P_5 + 2K_6 Q_5 \right) + U_{20} \left(-K_6 Q_6 + H_6 P_6 \right)$$

$$\frac{\partial \psi_2}{\partial Q_6} = U_{16} \left(-2H_5 P_6 + 2K_5 Q_6 \right) + U_{17} \left(-K_5 Q_5 + H_5 P_5 \right)$$
$$\qquad (29)$$
$$+U_{19} \left(-2H_6 P_6 + 2K_6 Q_6 \right) + U_{20} \left(-K_6 Q_5 + H_6 P_5 \right)$$

whence

$$\frac{dP_5}{dt} = \frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[-\frac{1}{4L_5} f_3^{(5,6)} Q_5 + \frac{1}{4\sqrt{L_5 L_6}} f_3^{(5,6)} Q_6 \right]$$
$$+ \left\{ -U_5 \left(2H_5 Q_5 + 2K_5 P_5 \right) + U_7 \left(K_5 P_6 + H_5 Q_6 \right) \right.$$
$$\left. -U_8 \left(2H_6 Q_5 + 2K_6 P_5 \right) + U_{10} \left(K_6 P_6 + H_6 Q_6 \right) \right\} \cos \left(5\lambda_6 - 2\lambda_5 \right) \quad (30)$$
$$+ \left\{ U_{15} \left(-2H_5 P_5 + 2K_5 Q_5 \right) + U_{17} \left(-K_5 Q_6 + H_5 P_6 \right) \right.$$
$$\left. +U_{18} \left(-2H_6 P_5 + 2K_6 Q_5 \right) + U_{20} \left(-K_6 Q_6 + H_6 P_6 \right) \right\} \sin(5\lambda_6 - 2\lambda_5)$$

$$\frac{dP_6}{dt} = \frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[-\frac{1}{4L_6} f_3^{(5,6)} Q_6 + \frac{1}{4\sqrt{L_5 L_6}} f_3^{(5,6)} Q_5 \right]$$
$$+ \left\{ -U_6 \left(2H_5 Q_6 + 2K_5 P_6 \right) + U_7 \left(K_5 P_5 + H_5 Q_5 \right) \right.$$
$$\left. -U_9 \left(2H_6 Q_6 + 2K_6 P_6 \right) + U_{10} \left(K_6 P_5 + H_6 Q_5 \right) \right\} \cos \left(5\lambda_6 - 2\lambda_5 \right) \quad (31)$$

$$+ \left\{ U_{16} \left(-2H_5 P_6 + 2K_5 Q_6 \right) + U_{17} \left(-K_5 Q_5 + H_5 P_5 \right) \right.$$

$$\left. + U_{19} \left(-2H_6 P_6 + 2K_6 Q_6 \right) + U_{20} \left(-K_6 Q_5 + H_6 P_5 \right) \right\} \sin(5\lambda_6 - 2\lambda_5)$$

Similarly:

$$\frac{dQ_u}{dt} \quad = \quad -\frac{\partial \psi_0}{\partial P_u} - \frac{\partial \psi_1}{\partial P_u} \cos\left(5\lambda_6 - 2\lambda_5 \right) - \frac{\partial \psi_2}{\partial P_u} \sin\left(5\lambda_6 - 2\lambda_5 \right) \tag{32}$$

$$-\frac{\partial \psi_0}{\partial P_5} \quad = \quad -\frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[-\frac{1}{4L_5} f_3^{(5,6)} P_5 + \frac{1}{4\sqrt{L_5 L_6}} f_3^{(5,6)} P_6 \right] \tag{33}$$

$$-\frac{\partial \psi_0}{\partial P_6} \quad = \quad -\frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[-\frac{1}{4L_6} f_3^{(5,6)} P_6 + \frac{1}{4\sqrt{L_5 L_6}} f_3^{(5,6)} P_5 \right] \tag{34}$$

$$-\frac{\partial \psi_1}{\partial P_5} \quad = \quad U_5 \left(-2H_5 P_5 + 2K_5 Q_5 \right) + U_7 \left(H_5 P_6 - K_5 Q_6 \right) + U_8 \left(2K_6 Q_5 - 2H_6 P_5 \right)$$

$$+ U_{10} \left(H_6 P_6 - K_6 Q_6 \right) \tag{35}$$

$$\frac{\partial \psi_1}{\partial P_6} \quad = \quad U_6 \left(2K_5 Q_6 - 2H_5 P_6 \right) + U_7 \left(H_5 P_5 - K_5 Q_5 \right) + U_9 \left(2K_6 Q_6 - 2H_6 P_6 \right)$$

$$+ U_{10} \left(H_6 P_5 - K_6 Q_5 \right) \tag{36}$$

$$-\frac{\partial \psi_2}{\partial P_5} \quad = \quad U_{15} \left(2K_5 P_5 + 2H_5 Q_5 \right) - U_{17} \left(K_5 P_6 + H_5 Q_6 \right) + U_{18} \left(2H_6 Q_5 + 2K_6 P_5 \right)$$

$$- U_{20} \left(K_6 P_6 + H_6 Q_6 \right) \tag{37}$$

$$-\frac{\partial \psi_2}{\partial P_6} \quad = \quad U_{16} \left(2K_5 P_6 + 2H_5 Q_6 \right) - U_{17} \left(K_5 P_5 + H_5 Q_5 \right) + U_{19} \left(2K_6 P_6 + 2H_6 Q_6 \right)$$

$$- U_{20} \left(K_6 P_5 + H_6 Q_5 \right) \tag{38}$$

Whence

$$\frac{dQ_5}{dt} \quad = \quad -\frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[-\frac{1}{4L_5} f_3^{(5,6)} P_5 + \frac{1}{4\sqrt{L_5 L_6}} f_3^{(5,6)} P_6 \right]$$

$$+ \{U_5 \left(2K_5Q_5 - 2H_5P_5\right) + U_7 \left(H_5P_6 - K_5Q_6\right)$$

$$+U_8 \left(2K_6Q_5 - 2H_6P_5\right) + U_{10} \left(H_6P_6 - K_6Q_6\right)\} \cos \left(5\lambda_6 - 2\lambda_5\right) \quad (39)$$

$$+ \{U_{15} \left(2K_5P_5 + 2H_5Q_5\right) - U_{17} \left(K_5P_6 + H_5Q_6\right)$$

$$+U_{18} \left(2K_6P_5 + 2H_6Q_5\right) - U_{20} \left(K_6P_6 + H_6Q_6\right)\} \sin(5\lambda_6 - 2\lambda_5)$$

$$\frac{dQ_6}{dt} = -\frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[-\frac{1}{4L_6} f_3^{(5,6)} P_6 + \frac{1}{4\sqrt{L_5 L_6}} f_3^{(5,6)} P_5 \right]$$

$$+ \{U_6 \left(2K_5Q_6 - 2H_5P_6\right) + U_7 \left(H_5P_5 - K_5Q_5\right)$$
$$+U_9 \left(2K_6Q_6 - 2H_6P_6\right) + U_{10} \left(H_6P_5 - K_6Q_5\right)\} \cos \left(5\lambda_6 - 2\lambda_5\right) \quad (40)$$

$$+ \{U_{16} \left(2K_5P_6 + 2H_5Q_6\right) - U_{17} \left(K_5P_5 + H_5Q_5\right)$$

$$+U_{19} \left(2K_6P_6 + 2H_6Q_6\right) - U_{20} \left(K_6P_5 + H_6Q_5\right)\} \sin(5\lambda_6 - 2\lambda_5)$$

The U's and the f's are functions in L_5, L_6.

We have for the variables L_u, λ_u, $u = 5$, 6.

$$\frac{dL_u}{dt} = \frac{\partial \psi_0}{\partial \lambda_u} + \frac{\partial}{\partial \lambda_u} \left[\{\psi_1 \cos \left(5\lambda_6 - 2\lambda_5\right) + \psi_2 \sin \left(5\lambda_6 - 2\lambda_5\right)\}\right] \quad (41)$$

$$\frac{dL_5}{dt} = \frac{2\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[\{U_1 \left(H_5^3 - 3H_5K_5^2\right) + U_2(H_6H_5^2 - H_6K_5^2\right.$$

$$-2H_5K_5K_6) + U_3 \left(H_5H_6^2 - H_5K_6^2 - 2H_6K_5K_6\right) + U_4 \left(H_6^3 - 3H_6K_6^2\right)$$

$$+U_5 \left(H_5P_5^2 - H_5Q_5^2 - 2K_5P_5Q_5\right) + U_6 \left(H_5P_6^2 - H_5Q_6^2 - 2K_5P_6Q_6\right)$$

$$+U_7 \left(K_5P_6Q_5 + K_5P_5Q_6 - H_5P_5P_6 + H_5Q_5Q_6\right) + U_8(H_6P_5^2 - H_6Q_5^2$$

$$-2K_6P_5Q_5) + U_9 \left(H_6P_6^2 - H_6Q_6^2 - 2K_6P_6Q_6\right) + U_{10}(K_6P_6Q_5$$

$$\left. +K_6P_5Q_6 - H_6P_5P_6 + H_6Q_5Q_6\right)\} \sin \left(5\lambda_6 - 2\lambda_5\right) \quad (42)$$

$$- \{U_{11} \left(K_5^3 - 3K_5H_5^2\right) + U_{12} \left(-K_6H_5^2 + K_6K_5^2 - 2H_5H_6K_5\right)$$

$$+ U_{13} \left(-K_5H_6^2 + K_5K_6^2 - 2H_5H_6K_6\right) + U_{14} \left(K_6^3 - 3K_6H_6^2\right)$$

$$+ U_{15} \left(-K_5P_5^2 + K_5Q_5^2 - 2H_5P_5Q_5\right) + U_{16} \left(K_5Q_6^2 - K_5P_6^2 - 2H_5P_6Q_6\right)$$

$$+ U_{17} \left(K_5P_5P_6 - K_5Q_5Q_6 + H_5P_6Q_5 + H_5P_5Q_6\right) + U_{18}(K_6Q_5^2 - K_6P_5^2$$

$$- 2H_6P_5Q_5) + U_{19} \left(K_6Q_6^2 - K_6P_6^2 - 2H_6P_6Q_6\right) + U_{20}(K_6P_5P_6$$

$$- K_6Q_5Q_6 + H_6Q_5P_6 + H_6P_5Q_6)\} \cos(5\lambda_6 - 2\lambda_5)]$$

$$\frac{dL_6}{dt} = \frac{5\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[- \{U_1 \left(H_5^3 - 3H_5K_5^2\right) + U_2(H_6H_5^2 - H_6K_5^2 \right.$$

$$- 2H_5K_5K_6) + U_3 \left(H_5H_6^2 - H_5K_6^2 - 2H_6K_5K_6\right) + U_4 \left(H_6^3 - 3H_6K_6^2\right)$$

$$+ U_5 \left(H_5P_5^2 - H_5Q_5^2 - 2K_5P_5Q_5\right) + U_6 \left(H_5P_6^2 - H_5Q_6^2 - 2K_5P_6Q_6\right)$$

$$+ U_7 \left(K_5P_6Q_5 + K_5P_5Q_6 - H_5P_5P_6 + H_5Q_5Q_6\right)$$

$$+ U_8 \left(H_6P_5^2 - H_6Q_5^2 - 2K_6P_5Q_5\right) + U_9 \left(H_6P_6^2 - H_6Q_6^2 - 2K_6P_6Q_6\right)$$

$$+ U_{10} \left(K_6P_6Q_5 + K_6P_5Q_6 - H_6P_5P_6 + H_6Q_5Q_6\right)\} \sin\left(5\lambda_6 - 2\lambda_5\right) \quad (43)$$

$$+ \{U_{11} \left(K_5^3 - 3K_5H_5^2\right) + U_{12} \left(-K_6H_5^2 + K_6K_5^2 - 2H_5H_6K_5\right)$$

$$+ U_{13} \left(-K_5H_6^2 + K_5K_6^2 - 2H_5H_6K_6\right) + U_{14} \left(K_6^3 - 3K_6H_6^2\right)$$

$$+ U_{15} \left(-K_5P_5^2 + K_5Q_5^2 - 2H_5P_5Q_5\right) + U_{16} \left(K_5Q_6^2 - K_5P_6^2 - 2H_5P_6Q_6\right)$$

$$+ U_{17} \left(K_5P_5P_6 - K_5Q_5Q_6 + H_5P_6Q_5 + H_5P_5Q_6\right)$$

$$+ U_{18} \left(K_6Q_5^2 - K_6P_5^2 - 2H_6P_5Q_5\right) + U_{19} \left(K_6Q_6^2 - K_6P_6^2 - 2H_6P_6Q_6\right)$$

$$+ U_{20} \left(K_6P_5P_6 - K_6Q_5Q_6 + H_6Q_5P_6 + H_6P_5Q_6\right)\} \cos\left(5\lambda_6 - 2\lambda_5\right)]$$

and

$$\frac{d\lambda_u}{dt} = -\frac{\partial \psi_0}{\partial L_u} - \frac{\partial \psi_1}{\partial L_u} \cos\left(5\lambda_6 - 2\lambda_5\right) - \frac{\partial \psi_2}{\partial L_u} \sin\left(5\lambda_6 - 2\lambda_5\right) \quad (44)$$

We find that:

$$\frac{\partial \psi_0}{\partial L_5} = -\frac{k^4 m_0^2 m_{05} \beta_5^3}{L_5^3} + \frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[\frac{1}{2} \frac{\partial f_1}{\partial L_5} + \frac{\left(L_5 \frac{\partial f_2}{\partial L_5} - f_2\right)}{8L_5^2} \left(H_5^2 + K_5^2\right) \right.$$

$$- \frac{\left(L_5 \frac{\partial f_3}{\partial L_5} - f_3\right)}{8L_5^2} \left(P_5^2 + Q_5^2\right) + \frac{1}{8L_6} \frac{\partial f_2}{\partial L_5} \left(H_6^2 + K_6^2\right) \qquad (45)$$

$$- \frac{1}{8L_6} \frac{\partial f_3}{\partial L_5} \left(P_6^2 + Q_6^2\right) + \frac{\left(2L_5 \frac{\partial f_3}{\partial L_5} - f_3\right)}{8L_5^{3/2} L_6^{1/2}} \left(P_5 P_6 + Q_5 Q_6\right)$$

$$\left. + \frac{\left(2L_5 \frac{\partial f_9}{\partial L_5} - f_9\right)}{8L_5^{3/2} L_6^{1/2}} \left(H_5 H_6 + K_5 K_6\right) \right]$$

$$\frac{\partial \psi_0}{\partial L_6} = -\frac{k^4 m_0^2 m_{06} \beta_6^3}{L_6^3} - \frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^3} \left[f_1 + \frac{f_2}{4L_5} \left(H_5^2 + K_5^2\right) \right.$$

$$- \frac{f_3}{4L_5} \left(P_5^2 + Q_5^2\right) + \frac{f_2}{4L_6} \left(H_6^2 + K_6^2\right) - \frac{f_3}{4L_6} \left(P_6^2 + Q_6^2\right)$$

$$\left. + \frac{f_3}{2 \left(L_5 L_6\right)^{1/2}} \left(P_5 P_6 + Q_5 Q_6\right) + \frac{f_9}{2 \left(L_5 L_6\right)^{1/2}} \left(H_5 H_6 + K_5 K_6\right) \right]$$

$$+ \frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[\frac{1}{2} \frac{\partial f_1}{\partial L_6} + \frac{1}{8L_5} \frac{\partial f_2}{\partial L_6} \left(H_5^2 + K_5^2\right) \qquad (46) \right.$$

$$- \frac{1}{8L_5} \frac{\partial f_3}{\partial L_6} \left(P_5^2 + Q_5^2\right) + \frac{\left(L_6 \frac{\partial f_2}{\partial L_6} - f_2\right)}{8L_6^2} \left(H_6^2 + K_6^2\right)$$

$$- \frac{\left(L_6 \frac{\partial f_3}{\partial L_6} - f_3\right)}{8L_6^2} \left(P_6^2 + Q_6^2\right) + \frac{\left(2L_6 \frac{\partial f_3}{\partial L_6} - f_3\right)}{8L_5^{1/2} L_6^{3/2}} \left(P_5 P_6 + Q_5 Q_6\right)$$

$$\left. + \frac{\left(2L_6 \frac{\partial f_9}{\partial L_6} - f_9\right)}{8L_5^{1/2} L_6^{3/2}} \left(H_5 H_6 + K_5 K_6\right) \right]$$

$$\frac{\partial \psi_1}{\partial L_5} = \frac{\partial U_1}{\partial L_5} \left(H_5^3 - 3H_5 K_5^2\right) + \frac{\partial U_2}{\partial L_5} \left(H_6 H_5^2 - H_6 K_5^2 - 2K_6 H_5 K_5\right)$$

$$+ \frac{\partial U_3}{\partial L_5} \left(H_5 H_6^2 - H_5 K_6^2 - 2K_5 H_6 K_6\right) + \frac{\partial U_4}{\partial L_5} \left(H_6^3 - 3H_6 K_6^2\right)$$

$$+ \frac{\partial U_5}{\partial L_5} \left(H_5 P_5^2 - H_5 Q_5^2 - 2K_5 P_5 Q_5\right) + \frac{\partial U_6}{\partial L_5} \left(H_5 P_6^2 - H_5 Q_6^2 - 2K_5 P_6 Q_6\right)$$

$$+ \frac{\partial U_7}{\partial L_5} \left(K_5 Q_5 P_6 + K_5 P_5 Q_6 - H_5 P_5 P_6 + H_5 Q_5 Q_6\right) \qquad (47)$$

$$+\frac{\partial U_8}{\partial L_5}\left(H_6P_5^2 - H_6Q_5^2 - 2K_6P_5Q_5\right) + \frac{\partial U_9}{\partial L_5}\left(H_6P_6^2 - H_6Q_6^2 - 2K_6P_6Q_6\right)$$

$$+\frac{\partial U_{10}}{\partial L_5}\left(K_6Q_5P_6 + K_6P_5Q_6 - H_6P_5P_6 + H_6Q_5Q_6\right)$$

$$\frac{\partial\psi_1}{\partial L_6} = \left[\frac{\partial U_1}{\partial L_6}\left(H_5^3 - 3H_5K_5^2\right) + \frac{\partial U_2}{\partial L_6}\left(H_6H_5^2 - H_6K_5^2 - 2K_6H_5K_5\right)\right.$$

$$+\frac{\partial U_3}{\partial L_6}\left(H_5H_6^2 - H_5K_6^2 - 2K_5H_6K_6\right) + \frac{\partial U_4}{\partial L_6}\left(H_6^3 - 3H_6K_6^2\right)$$

$$+\frac{\partial U_5}{\partial L_6}\left(H_5P_5^2 - H_5Q_5^2 - 2K_5P_5Q_5\right)$$

$$+\frac{\partial U_6}{\partial L_6}\left(H_5P_6^2 - H_5Q_6^2 - 2K_5P_6Q_6\right)\qquad(48)$$

$$+\frac{\partial U_7}{\partial L_6}\left(K_5Q_5P_6 + K_5P_5Q_6 - H_5P_5P_6 + H_5Q_5Q_6\right)$$

$$+\frac{\partial U_8}{\partial L_6}\left(H_6P_5^2 - H_6Q_5^2 - 2K_6P_5Q_5\right)$$

$$+\frac{\partial U_9}{\partial L_6}\left(H_6P_6^2 - H_6Q_6^2 - 2K_6P_6Q_6\right)$$

$$+\left.\frac{\partial U_{10}}{\partial L_6}\left(K_6Q_5P_6 + K_6P_5Q_6 - H_6P_5P_6 + H_6Q_5Q_6\right)\right]$$

$$\frac{\partial\psi_2}{\partial L_5} = \left[\frac{\partial U_{11}}{\partial L_5}\left(K_5^3 - 3K_5H_5^2\right) + \frac{\partial U_{12}}{\partial L_5}\left(K_6K_5^2 - K_6H_5^2 - 2H_6H_5K_5\right)\right.$$

$$+\frac{\partial U_{13}}{\partial L_5}\left(K_5K_6^2 - K_5H_6^2 - 2H_5H_6K_6\right) + \frac{\partial U_{14}}{\partial L_5}\left(K_6^3 - 3K_6H_6^2\right)$$

$$+\frac{\partial U_{15}}{\partial L_5}\left(K_5Q_5^2 - K_5P_5^2 - 2H_5P_5Q_5\right)$$

$$+\frac{\partial U_{16}}{\partial L_5}\left(K_5Q_6^2 - K_5P_6^2 - 2H_5P_6Q_6\right)\qquad(49)$$

$$+\frac{\partial U_{17}}{\partial L_5}\left(K_5P_5P_6 - K_5Q_5Q_6 + H_5Q_5P_6 + H_5P_5Q_6\right)$$

$$+\frac{\partial U_{18}}{\partial L_5}\left(K_6Q_5^2 - K_6P_5^2 - 2H_6P_5Q_5\right)$$

$$+\frac{\partial U_{19}}{\partial L_5}\left(K_6Q_6^2 - K_6P_6^2 - 2H_6P_6Q_6\right)$$

$$+\left.\frac{\partial U_{20}}{\partial L_5}\left(K_6P_5P_6 + H_6Q_5P_6 + H_6P_5Q_6 - K_6Q_5Q_6\right)\right]$$

$$
\begin{aligned}
\frac{\partial \psi_2}{\partial L_6} \quad = \quad & \left[\frac{\partial U_{11}}{\partial L_6} \left(K_5^3 - 3K_5 H_5^2 \right) + \frac{\partial U_{12}}{\partial L_6} \left(K_6 K_5^2 - K_6 H_5^2 - 2H_6 H_5 K_5 \right) \right. \\[2mm]
& + \frac{\partial U_{13}}{\partial L_6} \left(K_5 K_6^2 - K_5 H_6^2 - 2H_5 H_6 K_6 \right) + \frac{\partial U_{14}}{\partial L_6} \left(K_6^3 - 3K_6 H_6^2 \right) \\[2mm]
& + \frac{\partial U_{15}}{\partial L_6} \left(K_5 Q_5^2 - K_5 P_5^2 - 2H_5 P_5 Q_5 \right) \\[2mm]
& + \frac{\partial U_{16}}{\partial L_6} \left(K_5 Q_6^2 - K_5 P_6^2 - 2H_5 P_6 Q_6 \right) \qquad\qquad (50) \\[2mm]
& + \frac{\partial U_{17}}{\partial L_6} \left(K_5 P_5 P_6 - K_5 Q_5 Q_6 + H_5 Q_5 P_6 + H_5 P_5 Q_6 \right) \\[2mm]
& + \frac{\partial U_{18}}{\partial L_6} \left(K_6 Q_5^2 - K_6 P_5^2 - 2H_6 P_5 Q_5 \right) \\[2mm]
& + \frac{\partial U_{19}}{\partial L_6} \left(K_6 Q_6^2 - K_6 P_6^2 - 2H_6 P_6 Q_6 \right) \\[2mm]
& \left. + \frac{\partial U_{20}}{\partial L_6} \left(K_6 P_5 P_6 + H_6 Q_5 P_6 + H_6 P_5 Q_6 - K_6 Q_5 Q_6 \right) \right]
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{d\lambda_5}{dt} \quad = \quad & \frac{k^4 m_0^2 m_{05} \beta_5^3}{L_5^3} - \frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[\frac{1}{2} \frac{\partial f_1}{\partial L_5} \right. \\[2mm]
& + \frac{\left(L_5 \frac{\partial f_2}{\partial L_5} - f_2 \right)}{8 L_5^2} \left(H_5^2 + K_5^2 \right) - \frac{\left(L_5 \frac{\partial f_3}{\partial L_5} - f_3 \right)}{8 L_5^2} \left(P_5^2 + Q_5^2 \right) \\[2mm]
& + \frac{1}{8 L_6} \frac{\partial f_2}{\partial L_5} \left(H_6^2 + K_6^2 \right) - \frac{1}{8 L_6} \frac{\partial f_3}{\partial L_5} \left(P_6^2 + Q_6^2 \right) \\[2mm]
& + \frac{\left(2 L_5 \frac{\partial f_3}{\partial L_5} - f_3 \right)}{8 L_6^{1/2} L_5^{3/2}} \left(P_5 P_6 + Q_5 Q_6 \right) \\[2mm]
& \left. + \frac{\left(2 L_5 \frac{\partial f_9}{\partial L_5} - f_9 \right)}{8 L_6^{1/2} L_5^{3/2}} \left(H_5 H_6 + K_5 K_6 \right) \right] \\[2mm]
& - \left[\frac{\partial U_1}{\partial L_5} \left(H_5^3 - 3H_5 K_5^2 \right) + \frac{\partial U_2}{\partial L_5} \left(H_6 H_5^2 - H_6 K_5^2 - 2K_6 H_5 K_5 \right) \right. \\[2mm]
& + \frac{\partial U_3}{\partial L_5} \left(H_5 H_6^2 - H_5 K_6^2 - 2K_5 H_6 K_6 \right) + \frac{\partial U_4}{\partial L_5} \left(H_6^3 - 3H_6 K_6^2 \right)
\end{aligned}
$$

$$+ \frac{\partial U_5}{\partial L_5} \left(H_5 P_5^2 - H_5 Q_5^2 - 2K_5 P_5 Q_5 \right)$$

$$+ \frac{\partial U_6}{\partial L_5} \left(H_5 P_6^2 - H_5 Q_6^2 - 2K_5 P_6 Q_6 \right)$$

$$+ \frac{\partial U_7}{\partial L_5} \left(K_5 Q_5 P_6 + K_5 P_5 Q_6 - H_5 P_5 P_6 + H_5 Q_5 Q_6 \right)$$

$$+ \frac{\partial U_8}{\partial L_5} \left(H_6 P_5^2 - H_6 Q_5^2 - 2K_6 P_5 Q_5 \right)$$

$$+ \frac{\partial U_9}{\partial L_5} \left(H_6 P_6^2 - H_6 Q_6^2 - 2K_6 P_6 Q_6 \right) \tag{51}$$

$$+ \frac{\partial U_{10}}{\partial L_5} \left(K_6 Q_5 P_6 + K_6 P_5 Q_6 - H_6 P_5 P_6 + H_6 Q_5 Q_6 \right) \Big] \cos(5\lambda_6 - 2\lambda_5)$$

$$- \Big[\frac{\partial U_{11}}{\partial L_5} \left(K_5^3 - 3K_5 H_5^2 \right) + \frac{\partial U_{12}}{\partial L_5} \left(K_6 K_5^2 - K_6 H_5^2 - 2H_6 H_5 K_5 \right)$$

$$+ \frac{\partial U_{13}}{\partial L_5} \left(K_5 K_6^2 - K_5 H_6^2 - 2H_5 H_6 K_6 \right) + \frac{\partial U_{14}}{\partial L_5} \left(K_6^3 - 3K_6 H_6^2 \right)$$

$$+ \frac{\partial U_{15}}{\partial L_5} \left(K_5 Q_5^2 - K_5 P_5^2 - 2H_5 P_5 Q_5 \right)$$

$$+ \frac{\partial U_{16}}{\partial L_5} \left(K_5 Q_6^2 - K_5 P_6^2 - 2H_5 P_6 Q_6 \right)$$

$$+ \frac{\partial U_{17}}{\partial L_5} \left(K_5 P_5 P_6 - K_5 Q_5 Q_6 + H_5 Q_5 P_6 + H_5 P_5 Q_6 \right)$$

$$+ \frac{\partial U_{18}}{\partial L_5} \left(K_6 Q_5^2 - K_6 P_5^2 - 2H_6 P_5 Q_5 \right)$$

$$+ \frac{\partial U_{19}}{\partial L_5} \left(K_6 Q_6^2 - K_6 P_6^2 - 2H_6 P_6 Q_6 \right)$$

$$+ \frac{\partial U_{20}}{\partial L_5} \left(K_6 P_5 P_6 + H_6 Q_5 P_6 + H_6 P_5 Q_6 - K_6 Q_5 Q_6 \right) \Big] \sin(5\lambda_6 - 2\lambda_5)$$

$$\frac{d\lambda_6}{dt} = \frac{k^4 m_0^2 m_{06} \beta_6^3}{L_6^3} + \frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^3} \Big[f_1 + \frac{f_2}{4L_5} \left(H_5^2 + K_5^2 \right)$$

$$- \frac{f_3}{4L_5} \left(P_5^2 + Q_5^2 \right) + \frac{f_2}{4L_6} \left(H_6^2 + K_6^2 \right) - \frac{f_3}{4L_6} \left(P_6^2 + Q_6^2 \right)$$

$$+ \frac{f_3}{2 \left(L_5 L_6 \right)^{1/2}} \left(P_5 P_6 + Q_5 Q_6 \right) + \frac{f_9}{2 \left(L_5 L_6 \right)^{1/2}} \left(H_5 H_6 + K_5 K_6 \right) \Big]$$

$$-\frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2}\left[\frac{1}{2}\frac{\partial f_1}{\partial L_6} + \frac{1}{8L_5}\frac{\partial f_2}{\partial L_6}\left(H_5^2 + K_5^2\right)\right.$$

$$-\frac{1}{8L_5}\frac{\partial f_3}{\partial L_6}\left(P_5^2 + Q_5^2\right) + \frac{\left(L_6\frac{\partial f_2}{\partial L_6} - f_2\right)}{8L_6^2}\left(H_6^2 + K_6^2\right)$$

$$-\frac{\left(L_6\frac{\partial f_3}{\partial L_6} - f_3\right)}{8L_6^2}\left(P_6^2 + Q_6^2\right) + \frac{\left(2L_6\frac{\partial f_3}{\partial L_6} - f_3\right)}{8L_5^{1/2}L_6^{3/2}}\left(P_5 P_6 + Q_5 Q_6\right)$$

$$+\frac{\left(2L_6\frac{\partial f_9}{\partial L_6} - f_9\right)}{8L_5^{1/2}L_6^{3/2}}\left(H_5 H_6 + K_5 K_6\right)\right] \tag{52}$$

$$-\left[\frac{\partial U_1}{\partial L_6}\left(H_5^3 - 3H_5 K_5^2\right) + \frac{\partial U_2}{\partial L_6}\left(H_6 H_5^2 - H_6 K_5^2 - 2K_6 H_5 K_5\right)\right.$$

$$+\frac{\partial U_3}{\partial L_6}\left(H_5 H_6^2 - H_5 K_6^2 - 2K_5 H_6 K_6\right) + \frac{\partial U_4}{\partial L_6}\left(H_6^3 - 3H_6 K_6^2\right)$$

$$+\frac{\partial U_5}{\partial L_6}\left(H_5 P_5^2 - H_5 Q_5^2 - 2K_5 P_5 Q_5\right) + \frac{\partial U_6}{\partial L_6}\left(H_5 P_6^2 - H_5 Q_6^2 - 2K_5 P_6 Q_6\right)$$

$$+\frac{\partial U_7}{\partial L_6}\left(K_5 Q_5 P_6 + K_5 P_5 Q_6 - H_5 P_5 P_6 + H_5 Q_5 Q_6\right)$$

$$+\frac{\partial U_8}{\partial L_6}\left(H_6 P_5^2 - H_6 Q_5^2 - 2K_6 P_5 Q_5\right) + \frac{\partial U_9}{\partial L_6}\left(H_6 P_6^2 - H_6 Q_6^2 - 2K_6 P_6 Q_6\right)$$

$$\left.+\frac{\partial U_{10}}{\partial L_6}\left(K_6 Q_5 P_6 + K_6 P_5 Q_6 - H_6 P_5 P_6 + H_6 Q_5 Q_6\right)\right]\cos\left(5\lambda_6 - 2\lambda_5\right)$$

$$-\left[\frac{\partial U_{11}}{\partial L_6}\left(K_5^3 - 3K_5 H_5^2\right) + \frac{\partial U_{12}}{\partial L_6}\left(K_6 K_5^2 - K_6 H_5^2 - 2H_6 H_5 K_5\right)\right.$$

$$+\frac{\partial U_{13}}{\partial L_6}\left(K_5 K_6^2 - K_5 H_6^2 - 2H_5 H_6 K_6\right) + \frac{\partial U_{14}}{\partial L_6}\left(K_6^3 - 3K_6 H_6^2\right)$$

$$+\frac{\partial U_{15}}{\partial L_6}\left(K_5 Q_5^2 - K_5 P_5^2 - 2H_5 P_5 Q_5\right) + \frac{\partial U_{16}}{\partial L_6}\left(K_5 Q_6^2 - K_5 P_6^2 - 2H_5 P_6 Q_6\right)$$

$$+\frac{\partial U_{17}}{\partial L_6}\left(K_5 P_5 P_6 - K_5 Q_5 Q_6 + H_5 Q_5 P_6 + H_5 P_5 Q_6\right)$$

$$+\frac{\partial U_{18}}{\partial L_6}\left(K_6 Q_5^2 - K_6 P_5^2 - 2H_6 P_5 Q_5\right) + \frac{\partial U_{19}}{\partial L_6}\left(K_6 Q_6^2 - K_6 P_6^2 - 2H_6 P_6 Q_6\right)$$

$$\left.+\frac{\partial U_{20}}{\partial L_6}\left(K_6 P_5 P_6 + H_6 Q_5 P_6 + H_6 P_5 Q_6 - K_6 Q_5 Q_6\right)\right]\sin\left(5\lambda_6 - 2\lambda_5\right)$$

The f's and the U's are functions in L_5, L_6.

2. Methods for the solution of the equations of motion

The system of differential equations of motion is reduced to the following secular linear set; when we neglect terms of degree higher than the second in the Poincare' variables H_u, K_u, P_u, Qu, $u = 5, 6$.

$$\frac{dH_5}{dt} = \rho_1 K_5 + \rho_2 K_6$$

$$\frac{dH_6}{dt} = \rho_3 K_5 + \rho_4 K_6$$

$$(53)$$

$$\frac{dK_5}{dt} = \rho_5 H_5 + \rho_6 H_6$$

$$\frac{dK_6}{dt} = \rho_7 H_5 + \rho_8 H_6$$

and

$$\frac{dP_5}{dt} = \rho_9 Q_5 + \rho_{10} Q_6$$

$$\frac{dP_6}{dt} = \rho_{11} Q_5 + \rho_{12} Q_6$$

$$(54)$$

$$\frac{dQ_5}{dt} = \rho_{13} P_5 + \rho_{14} P_6$$

$$\frac{dQ_6}{dt} = \rho_{15} P_5 + \rho_{16} P_6$$

The ρ's are functions of L_5, L_6.

Evidently there exists for this set (53) and (54) the analytical, well known solution of Laplace – Lagrange for secular perturbations. But when we take into consideration the non – linear terms of the R.H.S. of Eqn. (4), we do not have a general analytical solution, it depends on the expression of the system. We can make some approximations to simplify the system of differential equations. We can from time to time find some solution or some approximation of the solution. By adding some hypothesis on the parameters of the system, we can have some solution as a series. There are numerical methods to resolve non linear systems (you can see it in many numerical analysis books). For every collection of numerical values of the parameters, we can search a numerical value of the solution, but we must be sure in advance that the solution is unique, near the particular values of the parameters, we have fixed. So, we can construct point by point the numerical solution. Alternatively, we may substitute the values of Hu, Ku, u, Q_u obtained from the secular perturbation equations (53), (54) in the R.H.S. of Eqs. (12), (13), (21), (22), (30), (31), (39), (40), (51), (52), as an approximation, to get the values of $\mathrm{d}(H_u, K_u, P_u, Q_u)/dt$ and repeat the process several times till we acquire the same numerical

results, namely the successive approximation method. Afterwards we integrate to find the values of $H_u(t)$, $K_u(t)$, $P_u(t)$, $Q_u(t)$ at any epoch. For the variables λ_u, we may use the formula:

$\lambda = n(t - t_0) + \tan^{-1}(\frac{-K}{H})$

where λ is the mean longitude, t_0 refer to the initial time and $n = \sqrt{\mu}\ a^{-3/2}$.

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