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A First order Jupiter Saturn Planetary Theory. Numerical Results

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We present the numerical analysis solution of the eight ordinary non linear differential equations of a first order secular J – S planetary theory. There is no general solution for these equations. We deal with the Poincare' variables H_u, K_u, P_u, Q_u; u=1,2 only. The solution is approximative, since we confine our treatment to a first order secular theory and truncate the Poisson series expansions at the fourth power in eccentricity – inclination.

Keywords: Celestial mechanics, Solar system dynamics, planetary theory.

1. Introduction

The aim of our article is to assign the first order secular approximate variations of the eccentricity e and the angular parameters i. ω, Ω, for the J – S subsystem, arising through the construction of a Hamiltonian canonical planetary theory. We take into account the J – S mutual perturbations only, and neglect the effect due to other planets. This is permissible since the J – S mutual perturbations represent 98% of the total one. So, it is a very nearly real problem. Our approach is an order by order one w.r.t. planetary masses O(m) and degree by degree w.r.t. e , $\gamma = \sin i$.

2. Theory

For the Jupiter Saturn first order secular planetary theory, the equations of motion in terms of Poincare‘ variables is given by:

$$\begin{aligned} \frac{dL_u}{dt} &= \frac{\partial F_{1s}}{\partial \lambda_u} \quad \frac{d\lambda_u}{dt} = -\frac{\partial F_{1s}}{\partial L_u} \quad \frac{dH_u}{dt} = \frac{\partial F_{1s}}{\partial K_u} \\ \frac{dK_u}{dt} &= -\frac{\partial F_{1s}}{\partial H_u} \quad \frac{dP_u}{dt} = \frac{\partial F_{1s}}{\partial Q_u} \quad \frac{dQ_u}{dt} = -\frac{\partial F_{1s}}{\partial P_u} \end{aligned} \tag{1}$$

$u = 1, 2$ (Script 1 refer to Jupiter, script 2 refer to Saturn).

Where, F_{1s} – the secular part of the Hamiltonian F reduced to its terms of order 0,1 in small parameter σ of the order of planetary masses.

$$F_{1s} = \frac{k^4 m_0^2 m_{01} \beta_1^3}{2L_1^2} + \frac{k^4 m_0^2 m_{02} \beta_2^3}{2L_2^2} + \frac{\sigma k^4 m_0 m_{02} \beta_1 \beta_2^3}{L_2^2} \left(\frac{a_2}{\Delta_{12}} \right)_s \tag{2}$$

with $a_2 = \frac{L_2^2}{k^2 m_0 m_{02} \beta_2^2}$

where $\left(\frac{a_2}{\Delta_{12}} \right)_s$ is given by:

$$\begin{aligned}
&\left(\frac{a_2}{\Delta_{12}} \right)_s = \frac{1}{2} f_1 + \frac{1}{8L_1} \left[f_2 \left(H_1^2 + K_1^2 \right) - f_3 \left(P_1^2 + Q_1^2 \right) \right] \\
&+ \frac{1}{8L_2} \left[f_2 \left(H_2^2 + K_2^2 \right) - f_3 \left(P_2^2 + Q_2^2 \right) \right] \\
&+ \frac{1}{128 L_1^2} [(f_6 - 4f_2) \left(H_1^2 + K_1^2 \right)^2 - 4 (2f_3 + f_5) \left(H_1^2 + K_1^2 \right) \left(P_1^2 + Q_1^2 \right) \\
&+ 3 f_4 \left(P_1^2 + Q_1^2 \right)^2 + 2 f_{14} \left\{ \left(H_1^2 - K_1^2 \right) \left(P_1^2 - Q_1^2 \right) + 4 H_1 K_1 P_1 Q_1 \right\}] \\
&+ \frac{1}{128 L_2^2} [(f_7 - 4f_2) \left(H_2^2 + K_2^2 \right)^2 - 4 (2f_3 + f_5) \left(H_2^2 + K_2^2 \right) \left(P_2^2 + Q_2^2 \right) \\
&+ 3 f_4 \left(P_2^2 + Q_2^2 \right)^2 + 2 f_{15} \left\{ \left(H_2^2 - K_2^2 \right) \left(P_2^2 - Q_2^2 \right) + 4 H_2 K_2 P_2 Q_2 \right\}] \\
&+ \frac{1}{4\sqrt{L_1 L_2}} \left[f_3 \left(P_1 P_2 + Q_1 Q_2 \right) + f_9 \left(H_1 H_2 + K_1 K_2 \right) \right] \\
&+ \frac{1}{32 L_1 \sqrt{L_1 L_2}} [- (f_3 + 3f_4) \left(P_1^2 + Q_1^2 \right) (P_1 P_2 + Q_1 Q_2) \\
&+ 2 (f_3 + f_5) \left(H_1^2 + K_1^2 \right) (P_1 P_2 + Q_1 Q_2) - f_{10} \left(P_1^2 + Q_1^2 \right) (H_1 H_2 + K_1 K_2) \\
&+ (f_{11} - f_9) \left(H_1^2 + K_1^2 \right) (H_1 H_2 + K_1 K_2) - f_{14} \{ \left(H_1^2 - K_1^2 \right) (P_1 P_2 - Q_1 Q_2) \\
&+ 2 H_1 K_1 (P_1 Q_2 + P_2 Q_1) \}
\end{aligned}$$

$$
\begin{aligned}
&+f_{16}\left\{\left(H_1H_2-K_1K_2\right)\left(P_1^2-Q_1^2\right)+2P_1Q_1\left(H_1K_2+H_2K_1\right)\right\}]\\
&+\frac{1}{32L_2\sqrt{L_1L_2}}[-\left(f_3+3f_4\right)\left(P_2^2+Q_2^2\right)\left(P_1P_2+Q_1Q_2\right)\\
&+2\left(f_3+f_5\right)\left(H_2^2+K_2^2\right)\left(P_1P_2+Q_1Q_2\right)\,-\,f_{10}\left(P_2^2+Q_2^2\right)\left(H_1H_2+K_1K_2\right)\\
&+\left(f_{12}-f_9\right)\left(H_2^2+K_2^2\right)\left(H_1H_2+K_1K_2\right)\\
&-f_{15}\left\{\left(H_2^2-K_2^2\right)\left(P_1P_2-Q_1Q_2\right)+2H_2K_2\left(P_1Q_2+P_2Q_1\right)\right\}\\
&+f_{16}\left\{\left(H_1H_2-K_1K_2\right)\left(P_2^2-Q_2^2\right)+2P_2Q_2\left(H_1K_2+H_2K_1\right)\right\}]\\
&+\frac{1}{64L_1L_2}[4f_3\left(P_1^2+Q_1^2\right)\left(P_2^2+Q_2^2\right)\\
&+3f_4\{\left(P_1^2+Q_1^2\right)\left(P_2^2+Q_2^2\right)+2(P_1P_2+Q_1Q_2)^2\}\\
&-2f_5\left\{\left(H_1^2+K_1^2\right)\left(P_2^2+Q_2^2\right)+\left(H_2^2+K_2^2\right)\left(P_1^2+Q_1^2\right)\right\}\\
&+2f_8\left(H_1^2+K_1^2\right)\left(H_2^2+K_2^2\right)\\
&+8f_9\left(H_2K_1-H_1K_2\right)\left(P_2Q_1-P_1Q_2\right)\\
&+4f_{10}\left(H_1H_2+K_1K_2\right)\left(P_1P_2+Q_1Q_2\right)\\
&+f_{13}\left\{\left(H_1^2-K_1^2\right)\left(H_2^2-K_2^2\right)+4H_1H_2K_1K_2\right\}\\
&+f_{14}\left\{\left(H_1^2-K_1^2\right)\left(P_2^2-Q_2^2\right)+4H_1K_1P_2Q_2\right\}\\
&+f_{15}\left\{\left(H_2^2-K_2^2\right)\left(P_1^2-Q_1^2\right)+4H_2K_2P_1Q_1\right\}\\
&-4f_{16}\{\left(H_1H_2-K_1K_2\right)\left(P_1P_2-Q_1Q_2\right)\\
&+\left(H_1K_2+H_2K_1\right)\left(P_1Q_2+P_2Q_1\right)\}]
\end{aligned}
\tag{3}
$$

Neglecting powers higher than the fourth in the Poincare‘ variables (H_1, H_2, K_1, K_2, $\ldots$, Q_1, Q_2).

Also, we can write $\left(\frac{a_2}{\alpha_{12}}\right)_s$, in the form:

$$
\begin{aligned}
&\left(\frac{a_2}{\Delta_{12}}\right)_s=A_1+A_2\left(H_1^2+K_1^2\right)+A_3\left(P_1^2+Q_1^2\right)+A_4\left(H_2^2+K_2^2\right)+A_5\left(P_2^2+Q_2^2\right)\\
&+A_6\left(H_1^2+K_1^2\right)^2+A_7\left(H_1^2+K_1^2\right)\left(P_1^2+Q_1^2\right)+A_8\left(P_1^2+Q_1^2\right)^2\\
&+A_9\left(H_1^2-K_1^2\right)\left(P_1^2-Q_1^2\right)+A_{10}H_1K_1P_1Q_1+A_{11}\left(H_2^2+K_2^2\right)^2\\
&+A_{12}\left(H_2^2+K_2^2\right)\left(P_2^2+Q_2^2\right)\;+A_{13}\left(P_2^2+Q_2^2\right)^2+A_{14}\left(H_2^2-K_2^2\right)\left(P_2^2-Q_2^2\right)\\
&+A_{15}H_2K_2P_2Q_2+A_{16}\left(P_1P_2+Q_1Q_2\right)+A_{17}\left(H_1H_2+K_1K_2\right)\\
&+A_{18}\left(P_1^2+Q_1^2\right)\left(P_1P_2+Q_1Q_2\right)+A_{19}\left(H_1^2+K_1^2\right)\left(P_1P_2+Q_1Q_2\right)\\
&+A_{20}\left(P_1^2+Q_1^2\right)\left(H_1H_2+K_1K_2\right)+A_{21}\left(H_1^2+K_1^2\right)\left(H_1H_2+K_1K_2\right)\\
&+A_{22}\left(H_1^2-K_1^2\right)\left(P_1P_2-Q_1Q_2\right)+A_{23}H_1K_1\left(P_1Q_2+P_2Q_1\right)\\
&+A_{24}\left(H_1H_2-K_1K_2\right)\left(P_1^2-Q_1^2\right)+A_{25}P_1Q_1\left(H_1K_2+H_2K_1\right)\\
&+A_{26}\left(P_2^2+Q_2^2\right)\left(P_1P_2+Q_1Q_2\right)+A_{27}\left(H_2^2+K_2^2\right)\left(P_1P_2+Q_1Q_2\right)
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
&+A_{28}\left(P_2^2+Q_2^2\right)(H_1H_2+K_1K_2)+A_{29}\left(H_2^2+K_2^2\right)(H_1H_2+K_1K_2)\\
&+A_{30}\left(H_2^2-K_2^2\right)(P_1P_2-Q_1Q_2)+A_{31}H_2K_2(P_1Q_2+P_2Q_1)\\
&+A_{32}(H_1H_2-K_1K_2)\left(P_2^2-Q_2^2\right)+A_{33}P_2Q_2(H_1K_2+H_2K_1)\\
&+A_{34}\left(P_1^2+Q_1^2\right)\left(P_2^2+Q_2^2\right)+A_{35}(P_1P_2+Q_1Q_2)^2\\
&+A_{36}\left\{\left(H_1^2+K_1^2\right)\left(P_2^2+Q_2^2\right)+\left(H_2^2+K_2^2\right)\left(P_1^2+Q_1^2\right)\right\}\\
&+A_{37}\left(H_1^2+K_1^2\right)\left(H_2^2+K_2^2\right)+A_{38}(H_1K_2-H_2K_1)\\
&+A_{39}(H_1H_2+K_1K_2)(P_1P_2+Q_1Q_2)+A_{40}\left(H_1^2-K_1^2\right)\left(H_2^2-K_2^2\right)\\
&+A_{41}H_1H_2K_1K_2+A_{42}\left(H_1^2-K_1^2\right)\left(P_2^2-Q_2^2\right)+A_{43}H_1K_1P_2Q_2\\
&+A_{44}\left(H_2^2-K_2^2\right)\left(P_1^2-Q_1^2\right)+A_{45}H_2K_2P_1Q_1\\
&+A_{46}\left\{(H_1H_2-K_1K_2)(P_1P_2-Q_1Q_2)+(H_1K_2+H_2K_1)(P_1Q_2+P_2Q_1)\right\}
\end{aligned}
$$

3. Numerical evaluations of Laplace coefficients and its derivatives (the f's)

We have,

$$
\begin{aligned}
&f_1=b_{\frac{1}{2}}^{(0)}=2.1795 \qquad f_2=2\alpha D_\alpha^1 b_{\frac{1}{2}}^{(0)}+\alpha^2 D_\alpha^2 b_{\frac{1}{2}}^{(0)}=1.7221\\
&f_3=\alpha b_{\frac{3}{2}}^{(1)}=1.7258 \qquad f_4=2\alpha^2 b_{\frac{5}{2}}^{(0)}+\alpha^2 b_{\frac{5}{2}}^{(2)}=10.8898\\
&f_5=2\alpha^2 D_\alpha^1 b_{\frac{3}{2}}^{(1)}+\alpha^3 D_\alpha^2 b_{\frac{3}{2}}^{(1)}=23.6278\\
&f_6=4\alpha^3 D_\alpha^3 b_{\frac{1}{2}}^{(0)}+\alpha^4 D_\alpha^4 b_{\frac{1}{2}}^{(0)}=14.1186\\
&f_7=24\alpha D_\alpha^1 b_{\frac{1}{2}}^{(0)}+36\alpha^2 D_\alpha^2 b_{\frac{1}{2}}^{(0)}+12\alpha^3 D_\alpha^3 b_{\frac{1}{2}}^{(0)}+\alpha^4 D_\alpha^4 b_{\frac{1}{2}}^{(0)}=70.1196\\
&f_8=4\alpha D_\alpha^1 b_{\frac{1}{2}}^{(0)}+14\alpha^2 D_\alpha^2 b_{\frac{1}{2}}^{(0)}+8\alpha^3 D_\alpha^3 b_{\frac{1}{2}}^{(0)}+\alpha^4 D_\alpha^4 b_{\frac{1}{2}}^{(0)}=28.7158\\
&f_9=2b_{\frac{1}{2}}^{(1)}-2\alpha D_\alpha^1 b_{\frac{1}{2}}^{(1)}-\alpha^2 D_\alpha^2 b_{\frac{1}{2}}^{(1)}=-1.1245\\
&f_{10}=2\alpha\left(b_{\frac{3}{2}}^{(0)}+b_{\frac{3}{2}}^{(2)}\right)-2\alpha^2\left(D_\alpha^1 b_{\frac{3}{2}}^{(0)}+D_\alpha^1 b_{\frac{3}{2}}^{(2)}\right)\\
&-\alpha^3\left(D_\alpha^2 b_{\frac{3}{2}}^{(0)}+D_\alpha^2 b_{\frac{3}{2}}^{(2)}\right)=-38.8956\\
&f_{11}=-4\alpha^2 D_\alpha^2 b_{\frac{1}{2}}^{(1)}-6\alpha^3 D_\alpha^3 b_{\frac{1}{2}}^{(1)}-\alpha^4 D_\alpha^4 b_{\frac{1}{2}}^{(1)}=-21.9616\\
&f_{12}=4b_{\frac{1}{2}}^{(1)}-4\alpha D_\alpha^1 b_{\frac{1}{2}}^{(1)}-22\alpha^2 D_\alpha^2 b_{\frac{1}{2}}^{(1)}-10\alpha^3 D_\alpha^3 b_{\frac{1}{2}}^{(1)}-\alpha^4 D_\alpha^4 b_{\frac{1}{2}}^{(1)}=-44.3355\\
&f_{13}=12b_{\frac{1}{2}}^{(2)}-12\alpha D_\alpha^1 b_{\frac{1}{2}}^{(2)}+6\alpha^2 D_\alpha^2 b_{\frac{1}{2}}^{(2)}+8\alpha^3 D_\alpha^3 b_{\frac{1}{2}}^{(2)}+\alpha^4 D_\alpha^4 b_{\frac{1}{2}}^{(2)}=25.6546\\
&f_{14}=6\alpha b_{\frac{3}{2}}^{(1)}+6\alpha^2 D_\alpha^1 b_{\frac{3}{2}}^{(1)}+\alpha^3 D_\alpha^2 b_{\frac{3}{2}}^{(1)}=51.8434\\
&f_{15}=2\alpha b_{\frac{3}{2}}^{(1)}-2\alpha^2 D_\alpha^1 b_{\frac{3}{2}}^{(1)}+\alpha^3 D_\alpha^2 b_{\frac{3}{2}}^{(1)}=9.2184\\
&f_{16}=2\alpha b_{\frac{3}{2}}^{(0)}-2\alpha^2 D_\alpha^1 b_{\frac{3}{2}}^{(0)}-\alpha^3 D_\alpha^2 b_{\frac{3}{2}}^{(0)}=-18.736
\end{aligned}
\tag{5}
$$

From [1], we have:
a_J – Semi – major axis of Jupiter = 5.202545 A.U.;
a_S – Semi – major axis of Saturn = 9.554841 A.U.;
$\alpha = \mathrm{a}_J/\mathrm{a}_S = 0.5445$, and the convergent series of the Laplace's coefficients is given by the formula:

$$\begin{aligned} \frac{1}{2} b_s^{(j)} &= \frac{s\,(s+1)\ldots(s+j-1)}{j!} \alpha^j \left\{ 1 + \frac{s\,(s+j)}{(j+1)} \alpha^2 \right. \\ &\quad \left. + \frac{s\,(s+1)\,(s+j)\,(s+j+1)}{2!\,(j+1)\,(j+2)} \alpha^4 + \ldots \right\} \ldots \end{aligned} \tag{6}$$

When $j = 0$, the factor outside of the brackets becomes unity. expanded up to α^{20} we get:

$$\begin{array}{lll} b_{\frac{1}{2}}^{(0)} = 2.1795 & b_{\frac{3}{2}}^{(0)} = 4.3404 & b_{\frac{5}{2}}^{(0)} = 13.4957 \\ b_{\frac{1}{2}}^{(1)} = 0.6194 & b_{\frac{3}{2}}^{(1)} = 3.1695 & b_{\frac{5}{2}}^{(1)} = 12.1949 \\ b_{\frac{1}{2}}^{(2)} = 0.2567 & b_{\frac{3}{2}}^{(2)} = 2.0691 & b_{\frac{5}{2}}^{(2)} = 9.7398 \\ D_\alpha^1 b_{\frac{1}{2}}^{(0)} = 0.8058 & D_\alpha^1 b_{\frac{3}{2}}^{(0)} = 14.4376 & D_\alpha^1 b_{\frac{1}{2}}^{(1)} = 1.4807 \\ D_\alpha^1 b_{\frac{3}{2}}^{(1)} = 15.0612 & D_\alpha^1 b_{\frac{1}{2}}^{(2)} = 1.1021 & D_\alpha^1 b_{\frac{3}{2}}^{(2)} = 13.2786 \\ D_\alpha^2 b_{\frac{1}{2}}^{(0)} = 2.8488 & D_\alpha^2 b_{\frac{3}{2}}^{(0)} = 92.3172 & D_\alpha^2 b_{\frac{1}{2}}^{(1)} = 2.5326 \\ D_\alpha^2 b_{\frac{3}{2}}^{(1)} = 91.0462 & D_\alpha^2 b_{\frac{1}{2}}^{(2)} = 3.5049 & D_\alpha^2 b_{\frac{3}{2}}^{(2)} = 90.0639 \\ D_\alpha^3 b_{\frac{1}{2}}^{(0)} = 11.6660 & D_\alpha^3 b_{\frac{1}{2}}^{(1)} = 12.5619 & D_\alpha^3 b_{\frac{1}{2}}^{(3)} = 12.6038 \\ D_\alpha^4 b_{\frac{1}{2}}^{(0)} = 74.9264 & D_\alpha^4 b_{\frac{1}{2}}^{(1)} = 77.2642 & D_\alpha^4 b_{\frac{1}{2}}^{(2)} = 82.6409\ldots \end{array} \tag{7}$$

Where: $D_\alpha^k = \frac{d^k}{d\alpha^k}$
We adopt the International Astronomical Union (IAU) system , i.e. we consider the units of length (the astronomical unit), mass (the mass of the Sun), and time (the day).
If the units of length, mass, and time are the astronomical units of these quantities then the astronomical unit of length is the length for which the Gaussian gravitational constant k has the value 0.0172029895, [1].
Also, from [1] we get the following values: m_J = mass of Jupiter = 1898.6×10^{24} kg
m_S = mass of Saturn = 568.46×10^{24}kg
m_0 = mass of Sun = 1.98911×10^{30}kg
$m_{0J} = \frac{m_0+m_1+\cdots+m_{j-1}}{m_0+m_1+\cdots+m_j}$
$m_{0J} = 0.9990$
$m_{0S} = 0.9997$
σ = small parameter = 10^{-3}
$\sigma\beta_J = \left(\frac{m_J}{m_0}\right)$ for Jupiter, $\beta_J = 0.9545$

$\sigma\beta_S = \left(\frac{m_S}{m_0}\right)$ for Saturn, $\beta_S = 0.2858$

$\mu_J = \frac{k^2 m_0}{m_{0J}} = 2.9612 \times 10^{-4}$ for Jupiter

$\mu_S = \frac{k^2 m_0}{m_{0S}} = 2.9592 \times 10^{-4}$ for Saturn

$L_J = \sqrt{\mu_J a_J} = 0.0393$ for Jupiter

$L_S = \sqrt{\mu_S} a_S = 0.0532$ for Saturn

$$\text{Common factor} = \frac{\alpha k^4 m_0 m_{0S} J_S^3}{L_S^2} = 6.8942\times10^{-10} \tag{8}$$

4. Numerical values of the A's

After some calculations, we get the following numerical values of the A's:

$$\begin{aligned}
&A_1=\frac{1}{2}f_1= 1.0898 A_2=\frac{f_2}{8L_1}= 5.4843 A_3=-\frac{f_3}{8L_1}=-5.4960\\
&A_4=\frac{f_2}{8L_2}= 4.0482 A_5=-\frac{f_3}{8L_2}=-4.0569\\
&A_6=\frac{f_6-4f_2}{128L_1^2}= 36.6651 A_7=-\frac{2f_3+f_5}{32L_1^2}=-549.2896\\
&A_8=\frac{3f_4}{128L_1^2}= 165.6701 A_9=\frac{f_{14}}{64L_1^2}= 525.8069 A_{10}=4A_9= 2.1032\times10^3\\
&A_{11}=\frac{f_7-4f_2}{128L_2^2}= 174.7099 A_{12}=-\frac{2f_3+f_5}{32L_2^2}=-299.2844\\
&A_{13}=\frac{3f_4}{128L_2^2}= 90.2665 A_{14}=\frac{f_{15}}{64L_2^2}= 50.9415\\
&A_{15}=4A_{14}= 203.7662 A_{16}=\frac{f_3}{4\sqrt{L_1L_2}}= 9.4439\\
&A_{17}=\frac{f_9}{4\sqrt{L_1L_2}}=-6.1536 A_{18}=-\frac{f_3+3f_4}{32L_1\sqrt{L_1L_2}}=-599.4198\\
&A_{19}=\frac{f_3+f_5}{16L_1\sqrt{L_1L_2}}= 883.6964 A_{20}=-\frac{f_{10}}{32L_1\sqrt{L_1L_2}}= 677.8514\\
&A_{21}=\frac{f_{11}-f_9}{32L_1\sqrt{L_1L_2}}=-363.1374 A_{22}=-\frac{f_{14}}{32L_1\sqrt{L_1L_2}}=-903.4975\\
&A_{23}=2A_{22}=-1.8070\times10^3 A_{24}=\frac{f_{16}}{32L_1\sqrt{L_1L_2}}=-326.5309\\
&A_{25}=2A_{24}=-653.0618 A_{26}=-\frac{f_3+3f_4}{32L_2\sqrt{L_1L_2}}=-442.4584\\
&A_{27}=\frac{f_3+f_5}{16L_2\sqrt{L_1L_2}}= 652.2956 A_{28}=-\frac{f_{10}}{32L_2\sqrt{L_1L_2}} = 500.3523
\end{aligned} \tag{9}$$

$$A_{29}{=}\frac{f_{12}{-}f_9}{32L_2\sqrt{L_1L_2}}{=}{-}555.8651 A_{30}{=}{-}\frac{f_{15}}{32L_2\sqrt{L_1L_2}}{=}{-}118.5855$$
$$A_{31}{=}2A_{30}{=}{-}237.1709 A_{32}{=}\frac{f_{16}}{32L_2\sqrt{L_1L_2}}{=}{-}241.0270$$
$$A_{33}{=}2A_{32}{=}{-}482.0540 A_{34}{=}\frac{4f_3{+}3f_4}{64L_1L_2}{=}\ 296.2563$$
$$A_{35}{=}\frac{3f_4}{32L_1L_2}{=}\ 489.1538 A_{36}{=}{-}\frac{f_5}{32L_1L_2}{=}{-}353.7756$$
$$A_{37}{=}\frac{f_8}{32L_1L_2}{=}\ 429.9574 A_{38}{=}\frac{f_9}{8L_1L_2}{=}{-}67.3484$$
$$A_{39}{=}\frac{f_{10}}{16L_1L_2}{=}{-}1.1648{\times}10^3 A_{40}{=}\frac{f_{13}}{64L_1L_2}{=}\ 192.0612$$
$$A_{41}{=}4A_{40}{=}\ 768.245 A_{42}{=}\frac{f_{14}}{64L_1L_2}{=}\ 388.1215$$
$$A_{43}{=}4A_{42}{=}\ 1.5525{\times}10^3 A_{44}{=}\frac{f_{15}}{64L_1L_2}{=}\ 69.0130$$
$$A_{45}{=}4A_{44}{=}\ 276.0519 A_{46}{=}{-}\frac{f_{16}}{16L_1L_2}\ {=}\ 561.0803.\ldots$$

The initial orbital elements for Jupiter and Saturn at J 2000, [1], are:
$e_{0j} = 0.04839266$, $i_{0j} = 1.30530^o$, $\alpha_{0j} = 14.75385^o$, $\alpha_{0j} = 100.55615^o$ for Jupiter;
$e_{0S} = 0.05415060$, $i_{0S} = 2.48446^o$, $\alpha_{0S} = 92.43194^o$, $\alpha_{0S} = 113.71504^o$ for Saturn.

Accordingly the numerical values of the initial Poincar variables for Jupiter and Saturn, at J 2000 are :

$$
\begin{aligned}
H_{0J} &= \sqrt{2L_J\left(1-\sqrt{1-e_{0J}^2}\right)}\cos\alpha_{0J} = 0.00927988\\
K_{0J} &= -sqrt2L_J\left(1-\sqrt{1-e_{0J}^2}\right)\sin\alpha_{0J} = -0.00244386\\
P_{0J} &= \sqrt{2L_J\sqrt{1-e_{0J}^2}\,(1-\cos i_{0J})}\cos\Omega_{0J} = -0.000826881\\
Q_{0J} &= -\sqrt{2L_J\sqrt{1-e_{0J}^2}\,(1-\cos i_{0J})}\sin\Omega_{0J}{=}-0.00443718\\
H_{0S} &= \sqrt{2L_S\left(1-\sqrt{1-e_{0S}^2}\right)}\cos\alpha_{0S} = -0.000530174 \qquad (10)\\
K_{0S} &= -sqrt2L_S\left(1-\sqrt{1-e_{0S}^2}\right)\sin\alpha_{0S} = -0.0124832\\
P_{0S} &= \sqrt{2L_S\sqrt{1-e_{0S}^2}\,(1-\cos i_{0S})}\cos\Omega_{0S} = -0.00401922\\
\Omega 0S &= -\sqrt{2L_S\sqrt{1-e_{0S}^2}\,(1-\cos i_{0S})}\sin\Omega_{0S}{=}-0.00914951\ldots
\end{aligned}
$$

5. The 8 first order nonlinear differential equations:

We have:

$$
\begin{aligned}
&\frac{dH_1}{dt}=\frac{\partial F_{1s}}{\partial K_1}=\frac{\sigma k^4 m_0 m_{021}\beta_2^3}{L_2^2}\{2A_2K_1+A_{17}K_2+4A_6K_1^3+A_{29}K_2^3+4A_6K_1H_1^2\\
&+A_{29}K_2H_2^2+(2A_7-2A_9)\,K_1P_1^2+(2A_7+2A_9)\,K_1Q_1^2+(A_{20}-A_{24})\,K_2P_1^2\\
&+(A_{20}+A_{24})\,K_2Q_1^2+3A_{21}K_1^2K_2+(A_{28}-A_{32})\,K_2P_2^2+(A_{28}+A_{32})\,K_2Q_2^2\\
&+(2A_{36}-2A_{42})\,K_1P_2^2+(2A_{36}+2A_{42})\,K_1Q_2^2+A_{21}K_2H_1^2\\
&+(2A_{37}-2A_{40})\,K_1H_2^2+(2A_{37}+2A_{40})\,K_1K_2^2+2A_{21}H_1H_2\ K_1\\
&+(2A_{19}-2A_{22})\,K_1P_1P_2+(2A_{19}+2A_{22})\,K_1Q_1Q_2+A_{41}H_1H_2K_2\\
&+A_{10}H_1P_1Q_1+A_{33}H_2P_2Q_2+(A_{38}+A_{46})\,H_2P_2Q_1\\
&-(A_{38}-A_{46})\,H_2P_1Q_2+(A_{39}+A_{46})\,K_2Q_1Q_2+(A_{39}-A_{46})\,K_2P_1P_2\\
&+A_{23}H_1P_1Q_2+A_{23}H_1P_2Q_1+A_{25}H_2P_1Q_1+A_{43}H_1P_2Q_2\}\ldots
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&\frac{dK_1}{dt}=-\frac{\partial F_{1s}}{\partial H_1}=-\frac{\sigma k^4 m_0 m_{02}\beta_1\beta_2^3}{L_2^2}\{2A_2H_1+A_{17}H_2+4A_6H_1^3+A_{29}H_2^3\\
&+4A_6H_1K_1^2+A_{29}H_2K_2^2+(2A_7+2A_9)\,H_1P_1^2+(2A_7-2A_9)\,H_1Q_1^2\\
&+(A_{20}+A_{24})\,H_2P_1^2+(A_{20}-A_{24})\,H_2Q_1^2+3A_{21}H_1^2H_2+(A_{28}-A_{32})\,H_2Q_2^2\\
&+(A_{28}+A_{32})\,H_2P_2^2+(2A_{36}-2A_{42})\,H_1Q_2^2+(2A_{36}+2A_{42})\,H_1P_2^2+A_{21}H_2K_1^2\\
&+(2A_{37}-2A_{40})\,H_1K_2^2+(2A_{37}+2A_{40})\,H_1H_2^2+2A_{21}H_1K_1K_2\\
&+(2A_{19}-2A_{22})\,H_1Q_1Q_2+(2A_{19}+2A_{22})\,H_1P_1P_2+A_{41}H_2K_1K_2+A_{10}K_1P_1Q_1\\
&+A_{33}K_2P_2Q_2-(A_{38}-A_{46})\,K_2P_2Q_1+(A_{38}+A_{46})\,K_2P_1Q_2\\
&+(A_{39}-A_{46})\,H_2Q_1Q_2+(A_{39}+A_{46})\,H_2P_1P_2+A_{23}K_1P_1Q_2+A_{23}K_1P_2Q_1\\
&+A_{25}K_2P_1Q_1+A_{43}K_1P_2Q_2\}
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&\frac{dH_2}{dt}=\frac{\partial F_{1s}}{\partial K_2}=\frac{\sigma k^4 m_0 m_{021}\beta_2^3}{L_2^2}\{2A_4K_2+A_{17}K_1+4A_{11}K_2^3+A_{21}K_1^3\\
&+4A_{11}H_2^2K_2+A_{21}H_1^2K_1+(2A_{12}-2A_{14})\,K_2P_2^2+(2A_{12}+2A_{14})\,K_2Q_2^2\\
&+(A_{20}-A_{24})\,K_1P_1^2+(A_{20}+A_{24})\,K_1Q_1^2+3A_{29}K_1K_2^2+(A_{28}-A_{32})\,K_1P_2^2\\
&+(A_{28}+A_{32})\,K_1Q_2^2+(2A_{36}-2A_{44})\,K_2P_1^2+(2A_{36}+2A_{44})\,K_2Q_1^2+A_{29}H_2^2K_1\\
&+(2A_{37}-2A_{40})\,H_1^2K_2+(2A_{37}+2A_{40})\,K_1^2K_2+2A_{29}H_1H_2K_2\\
&+(2A_{27}-2A_{30})\,K_2P_1P_2+(2A_{27}+2A_{30})\,K_2Q_1Q_2+A_{41}H_1H_2K_1+A_{15}H_2P_2Q_2\\
&+A_{25}H_1P_1Q_1+(A_{38}+A_{46})\,H_1P_1Q_2-(A_{38}-A_{46})\,H_1P_2Q_1\\
&+(A_{39}+A_{46})\,K_1Q_1Q_2+(A_{39}-A_{46})\,K_1P_1P_2+A_{31}H_2P_1Q_2+A_{31}H_2P_2Q_1\\
&+A_{45}H_2P_1Q_1+A_{33}H_1P_2Q_2\}\ldots
\end{aligned}
\tag{13}
$$

$$\frac{dK_2}{dt}=-\frac{\partial F_{1s}}{\partial H_2}=-\frac{\sigma k^4 m_0 m_{02}\beta_1\beta_2^3}{L_2^2}\{2A_4H_2+A_{17}H_1+4A_{11}H_2^3+A_{21}H_1^3$$
$$+4A_{11}H_2K_2^2+A_{21}H_1K_1^2+(2A_{12}+2A_{14})H_2P_2^2$$
$$+(2A_{12}-2A_{14})H_2Q_2^2+(A_{20}+A_{24})H_1P_1^2$$
$$+(A_{20}-A_{24})H_1Q_1^2+3A_{29}H_1H_2^2+(A_{28}+A_{32})H_1P_2^2$$
$$+(A_{28}-A_{32})H_1Q_2^2+(2A_{36}+2A_{44})H_2P_1^2$$
$$+(2A_{36}-2A_{44})H_2Q_1^2+A_{29}H_1K_2^2+(2A_{37}-2A_{40})H_2K_1^2$$
$$+(2A_{37}+2A_{40})H_1^2H_2+2A_{29}H_2K_1K_2 \tag{14}$$
$$+(2A_{27}+2A_{30})H_2P_1P_2+(2A_{27}-2A_{30})H_2Q_1Q_2$$
$$+A_{41}H_1K_1K_2+A_{15}K_2P_2Q_2+A_{25}K_1P_1Q_1$$
$$-(A_{38}-A_{46})K_1P_1Q_2+(A_{38}+A_{46})K_1P_2Q_1$$
$$+(A_{39}-A_{46})H_1Q_1Q_2+(A_{39}+A_{46})H_1P_1P_2+A_{31}K_2P_1Q_2$$
$$+A_{31}K_2P_2Q_1+A_{45}K_2P_1Q_1+A_{33}K_1P_2Q_2\}\dots$$

$$\frac{dP_1}{dt}=\frac{\partial F_{1s}}{\partial Q_1}=\frac{\sigma k^4 m_0 m_{02}\beta_1\beta_2^3}{L_2^2}\{2A_3Q_1+A_{16}Q_2+4A_8Q_1^3+A_{26}Q_2^3$$
$$+4A_8P_1^2Q_1+A_{10}H_1K_1P_1+(2A_7-2A_9)H_1^2Q_1$$
$$+(2A_7+2A_9)K_1^2Q_1+(A_{19}-A_{22})H_1^2Q_2$$
$$+(A_{19}+A_{22})K_1^2Q_2+3A_{18}Q_1^2Q_2+(2A_{20}-2A_{24})H_1H_2Q_1$$
$$+(2A_{20}+2A_{24})K_1K_2Q_1+(2A_{36}-2A_{44})H_2^2Q_1$$
$$+(2A_{36}+2A_{44})K_2^2Q_1+A_{18}P_1^2Q_2+(2A_{34}+2A_{35})Q_1Q_2^2$$
$$+2A_{18}P_1P_2Q_1+A_{23}H_1K_1P_2+(A_{27}-A_{30})H_2^2Q_2 \tag{15}$$
$$+(A_{27}+A_{30})K_2^2Q_2+A_{26}P_2^2Q_2+2A_{34}P_2^2Q_1+A_{25}H_1K_2P_1$$
$$+(A_{38}+A_{46})H_2K_1P_2-(A_{38}-A_{46})H_1K_2P_2$$
$$+(A_{39}+A_{46})K_1K_2Q_2+(A_{39}-A_{46})H_1H_2Q_2$$
$$+A_{31}H_2K_2P_2+2A_{35}P_1P_2Q_2+A_{45}H_2K_2P_1+A_{25}H_2K_1P_1\}\dots$$

$$\frac{dQ_1}{dt}=-\frac{\partial F_{1s}}{\partial P_1}=-\frac{\sigma k^4 m_0 m_{02}\beta_1\beta_2^3}{L_2^2}\{2A_3P_1+A_{16}P_2+4A_8P_1^3+A_{26}P_2^3$$
$$+(2A_7+2A_9)H_1^2P_1+(2A_7-2A_9)K_1^2P_1+4A_8P_1Q_1^2$$
$$+3A_{18}P_1^2P_2+A_{10}H_1K_1Q_1+(A_{19}+A_{22})H_1^2P_2$$
$$+(A_{19}-A_{22})K_1^2P_2+(2A_{20}+2A_{24})H_1H_2P_1+(2A_{20}-2A_{24})K_1K_2P_1$$
$$+(A_{27}+A_{30})H_2^2P_2+(A_{27}-A_{30})K_2^2P_2+2A_{18}P_1Q_1Q_2+A_{18}P_2Q_1^2 \tag{16}$$
$$+A_{23}H_1K_1Q_2+A_{25}H_1K_2Q_1+A_{25}H_2K_1Q_1+A_{26}P_2Q_2^2+A_{31}H_2K_2Q_2$$
$$+(2A_{34}+2A_{35})P_1P_2^2+(2A_{36}+2A_{44})H_2^2P_1+(2A_{36}-2A_{44})K_2^2P_1$$
$$+2A_{34}P_1Q_2^2+2A_{35}P_2Q_1Q_2+(A_{38}+A_{46})H_1K_2Q_2-(A_{38}-A_{46})H_2K_1Q_2$$
$$+(A_{39}+A_{46})H_1H_2P_2+(A_{39}-A_{46})K_1K_2P_2+A_{45}H_2K_2Q_1\}\dots$$

$$\begin{aligned}\frac{dP_2}{dt}&=\frac{\partial F_{1s}}{\partial Q_2}=\frac{\sigma k^4 m_0 m_{02}\beta_1\beta_2^3}{L_2^2}\{2A_5Q_2+A_{16}Q_1+4A_{13}Q_2^3+A_{15}H_2K_2P_2\\&+A_{26}P_2^2Q_1+A_{18}Q_1^3+4A_{13}P_2^2Q_2+(2A_{12}-2A_{14})H_2^2Q_2\\&+(2A_{12}+2A_{14})K_2^2Q_2+A_{18}P_1^2Q_1+(A_{19}-A_{22})H_1^2Q_1\\&+(A_{19}+A_{22})K_1^2Q_1+3A_{26}Q_1Q_2^2+(A_{27}-A_{30})H_2^2Q_1\\&+(A_{27}+A_{30})K_2^2Q_1+A_{23}H_1K_1P_1+2A_{26}P_1P_2Q_2\\&+(2A_{28}-2A_{32})H_1H_2Q_2+(2A_{28}+2A_{32})K_1K_2Q_2+A_{31}H_2K_2P_1\\&+A_{33}H_1K_2P_2+A_{33}H_2K_1P_2+(2A_{34}+2A_{35})Q_1^2Q_2\\&+2A_{34}P_1^2Q_2+2A_{35}P_1P_2Q_1+(2A_{36}-2A_{42})H_1^2Q_2\\&+(2A_{36}+2A_{42})K_1^2Q_2+(A_{38}+A_{46})H_1K_2P_1-(A_{38}-A_{46})H_2K_1P_1\\&+A_{43}H_1K_1P_2+(A_{39}-A_{46})H_1H_2Q_1+(A_{39}+A_{46})K_1K_2Q_1\}\ldots\end{aligned}\tag{17}$$

$$\begin{aligned}\frac{dQ_2}{dt}&=-\frac{\partial F_{1s}}{\partial P_2}=-\frac{\sigma k^4 m_0 m_{02}\beta_1\beta_2^3}{L_2^2}\{2A_5P_2+A_{16}P_1+4A_{13}P_2^3+A_{15}H_2K_2Q_2\\&+A_{26}P_1Q_2^2+A_{18}P_1^3+4A_{13}P_2Q_2^2+(2A_{12}-2A_{14})K_2^2P_2\\&+(2A_{12}+2A_{14})H_2^2P_2+A_{18}P_1Q_1^2+(A_{19}-A_{22})K_1^2P_1\\&+(A_{19}+A_{22})H_1^2P_1+3A_{26}P_1P_2^2+(A_{27}-A_{30})K_2^2P_1\\&+(A_{27}+A_{30})H_2^2P_1+A_{23}H_1K_1Q_1+2A_{26}P_2Q_1Q_2+(2A_{28}-2A_{32})K_1\\&K_2P_2+(2A_{28}+2A_{32})H_1H_2P_2+A_{31}H_2K_2Q_1+A_{33}H_1K_2Q_2+A_{33}H_2K_1Q_2\\&+(2A_{34}+2A_{35})P_1^2P_2+2A_{34}P_2Q_1^2+2A_{35}P_1Q_1Q_2+(2A_{36}-2A_{42})K_1^2P_2\\&+(2A_{36}+2A_{42})H_1^2P_2+(A_{38}+A_{46})H_2K_1Q_1-(A_{38}-A_{46})H_1K_2Q_1\\&+A_{43}H_1K_1Q_2+(A_{39}-A_{46})K_1K_2P_1+(A_{39}+A_{46})H_1H_2P_1\}\ldots\end{aligned}\tag{18}$$

6. Numerical integration

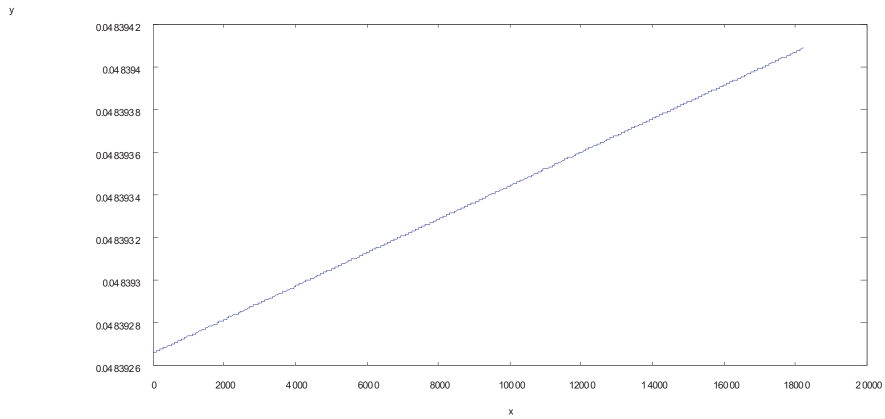
We need a numerical solution, of the eight differential equations (11–18), which verify the numerical values, given by [10], of the initial Poincar variables for Jupiter and Saturn, at J 2000. So we write a suitable calculation program (a MACSYMA program) using the fourth order Runge – Kutta method. The calculations are made in double precision. The unit of time is the day, the total integration time is 18250 days (≈50 years), so the time interval of integration is [0, 18250], the step of time is $\triangle$t = 1 day, with $t_1 = 0 \equiv$ J2000, …, $t_i = (i-1)\triangle t$, $i = 1, 2, \ldots, 18250$.

By this integration, we obtain the numerical values of the Poincar variables : $[(H_1)_i, (K_1)_i, (H_2)_i, (K_2)_i, (P1)_i, (Q_1)_i, (P_2)_i, (Q_2)_i]$, for $i = 1, \ldots, 18250$. Then, we deduce, the corresponding values of orbit parameters for Jupiter and Saturn:
$[(e_J)_i, (i_J)_i, (\alpha_J)_i, (\alpha_J)_i, (e_S)_i, (i_S)_i, (\alpha_S)_i, (\alpha_{Si}]$, $i = 1, 2, \ldots, 18250$.

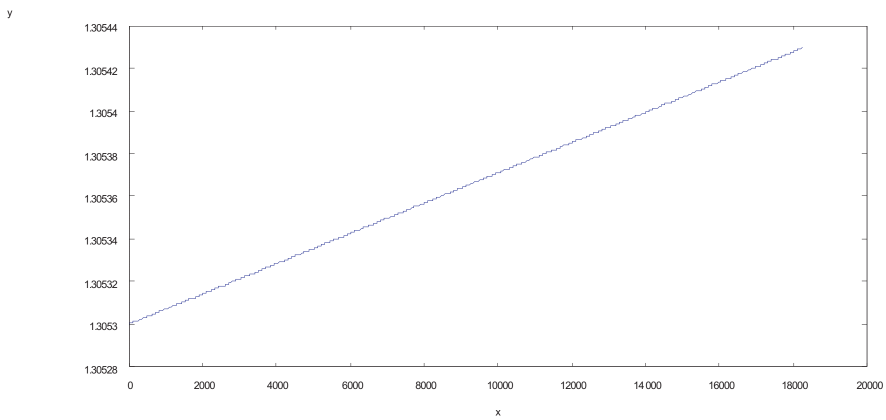
It is obvious that this huge number of numerical values cannot be included in this paper, so we give merely the graphs of these parameters. These graphs are made by using only one point every 10 days:

$$[t_{10.i},\ (e_J)_{10i}], [t_{10.i},\ (i_J)_{10i}],\ \ldots,\ i = 1, \ldots, 1825$$

The eccentricity of Jupiter :

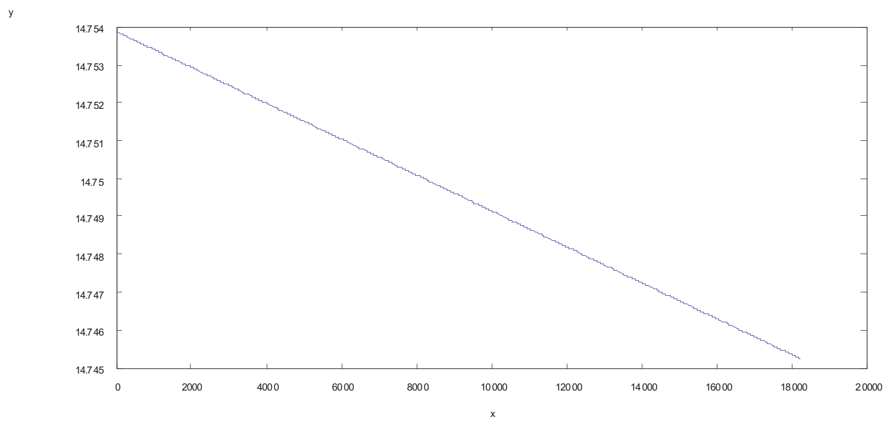


It is increasing very slowly from 0.048392658470653 to 0.048394091346053.
Inclination of Jupiter in degrees:



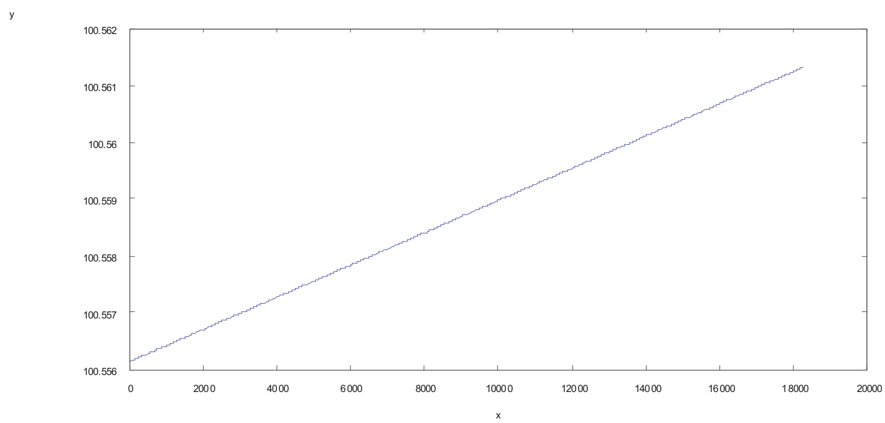
It is increasing from $1^{\circ}.305299975526787$ to $1^{\circ}.305429659328161$.

Graph of α of Jupiter :



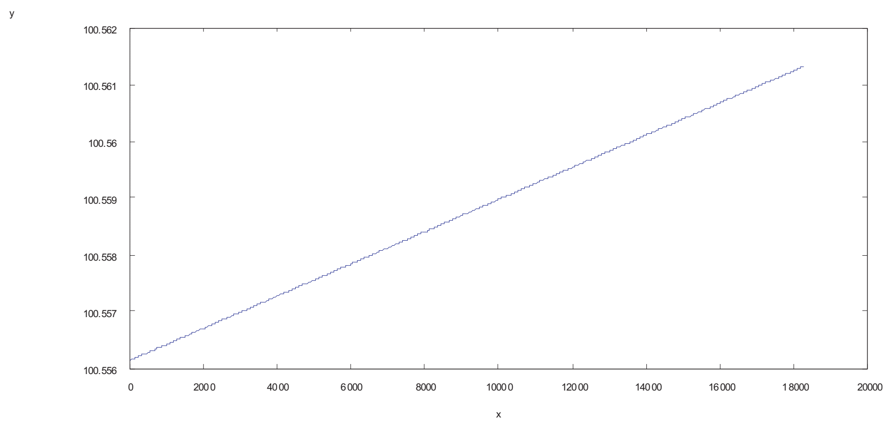
It is decreasing from $14^{\circ}.75387282599478$ to $14^{\circ}.74522945392956$.

Graph of Ω of Jupiter:



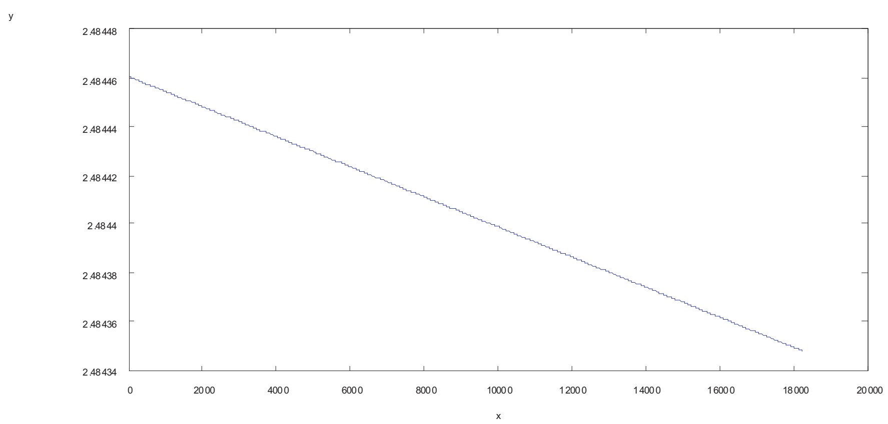
It is increasing from $100^{\circ}.556135827009$ to $100^{\circ}.561313056631$.

Eccentricity of Saturn:



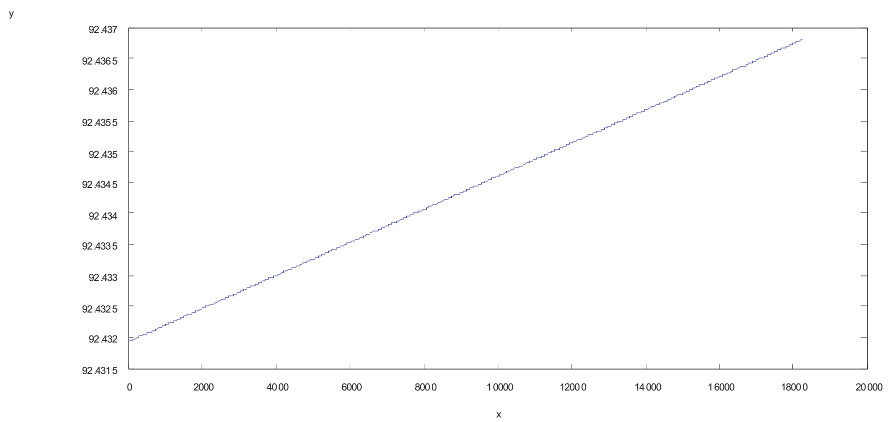
It is increasing from 0.054150453249948 to 0.054154043521506.

Inclination of Saturn:



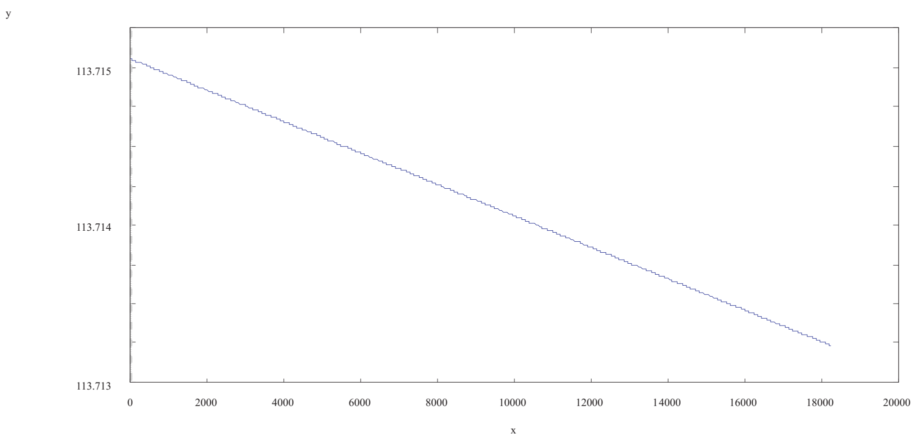
It is decreasing from $2^{\circ}.484460574452989$ to $2^{\circ}.484347881368836$.

Graph of α of Saturn:



It is increasing from 92°.43193165331819 to 92°.43680397027501.

Graph of Ω of Saturn:



The angle Ω of Saturn is decreasing from 113°.7150448346539 to 113°.7136515149878.

7. Discussion

We have three categories for handling the problem: the analytical, the numerical and the semi – analytical. Our approach is a NEW semi – analytical one, which is recommended by celestial mechanicians. The analytical techniques are of the same accuracy as the numerical integration procedures. The period of variation had been extended to hundreds of millions, or even more, by other investigators, using numerical integration and analytical techniques (BDL group: Chapront, Bretagnon, Simon, ...– Italian group: Milani, Nobili, Carpino, ...– A.E. Roy's group: LONGSTOP project). The French Egyptian group: Meffroy, Kamel, Soliman, ... extended successfully the analytical technique of the problem intensively, to very high orders w.r.t. masses and to rather high degrees in the eccentricity – inclination, but without numerical results. The period of variations of H_u, K_u, P_u, Q_u is 50 years only. This can be expanded easily to a longer interval of time and also, we can easily include numerical values of more than four decimals. It is the first time to produce numerical results relevant to Meffroy – Kamel analytical planetary theory. The ephemeris for an interval of 50 yrs., could be requested from the authors.

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