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Computing Simulation of the Generalized Duffing Oscillator Basedon EBM and MHPM

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This paper is concerned with analytical approximate solutions, to the generalized Duffing oscillation. Modified Homotopy Perturbation Method (MHPM) and Energy Balance Method (EBM) are applied to solve nonlinear equation and consequently the relationship between the natural frequency and the initial amplitude is obtained in an analytical form.

The general solution can be used to yield the relationship between amplitude and frequency in different strengths of nonlinearity. To verify the accuracy of the present approach, illustrative examples are provided and compared with exact solutions. The procedure yields rapid convergence with respect to the exact solution obtained by numerical integration.

Keywords: nonlinear oscillation, generalized Duffing equation, Modified Homotopy Perturbation Method, Energy Balance Method.

1. Introduction

Duffing equation is a mathematical model for describing a classical oscillator driven by a periodic force and it has been studied by many researchers because of its widely applied background [1–4]. In an earlier paper, Pirbodaghi, et al. [5] used Homotopy Analysis Method (HAM) and Homotopy Pade' Technique to obtain an accurate analytical solution for Duffing equations with cubic and quintic Nonlinearities. Beléndez, et al. [6] calculated analytical approximations to the periodic solutions to the quintic Duffing oscillator. In another study [7] He's Energy Balance Method (EBM) and Amplitude Frequency Formulation (AFF) was employed by Turgut Ozis and Ahmet Yildirim to construct the frequency–amplitude relation for a Duffing–harmonic oscillator.

Most of the phenomena in engineering such as Duffing oscillators are essentially nonlinear. Because of the difficulties of solving nonlinear equations, using helpful and simple approaches are very important. In the past decades, Many asymptotic techniques including the Energy Balance method (EBM) [8], Hamiltonian Approach (HA) [9], the Max–Min approach (MMA) [10], Parameter Expansion Method (PEM) [11], Homotopy perturbation method (HPM) [12–14], Variational iteration method (VIM) [15–17]and Differential transformation method (DTM) [18, 19] have been developed to construct many types of exact solutions of nonlinear ordinary differential equation.

In this work, the following Duffing equation is considered:

$$\begin{aligned} &\ddot{u} + u + a_3 u^3 + a_5 u^5 + \cdots + a_{2q-1} u^{2q-1} \\ &q = \{k | k \leq 2 \qquad k \in N\} \end{aligned} \tag{1}$$

With initial conditions: $u(0) = A, \dot{u}(0) = 0$.

The organization of this paper is as follows. The basic concepts of MHPM and EBM are explained in section 2, Application of solution procedure illustrated in both modified homotopy perturbation method and Energy balance method in section 3. In Section 4, three special cases of the generalized Duffing oscillator are considered and the analytic results obtained from the present study are compared with those from the exact solution, followed by a conclusion in Section 5.

2. Basic concept

Basic ideas of modified homotopy perturbation method and Energy balance method are considered and explained as follows:

2.1. *Modified homotopy perturbation method*

To describe the basic concept of this method, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0 \qquad r \in \Omega \tag{2}$$

Subject to boundary condition:

$$B(u, \delta u/\delta n) = 0 \qquad r \in \Gamma \tag{3}$$

where $A, B, f(r)$ and Γ are a general differential operator, a boundary operator, a known analytical function, and the boundary of domain Ω.

Generally speaking the operator Acan divided into a linear part L and a nonlinear part$N(u)$. Equation (2) can so, be rewritten as:

$$L(u) + N(u) - f(r) \tag{4}$$

We construct a homotopy of equation (2) $\nu(r, p) : \Omega \times [0, 1] \to R$ which satisfied Equation (5):

$$H(\nu, p) = L(\nu) - L(u_0) + pL(u_0) + p[N(\nu) - f(r)] = 0 \tag{5}$$

where p is embedding parameter and u_0 is an initial guess approximation of equation (2) which satisfies the boundary condition.

According to Modified homotopy perturbation method, the solution is expanded into series of p in the following form:

$$u = \sum_{i=1}^{n} p^i u_i \tag{6}$$

Frequency is expanded in similar way:

$$1 = \omega^2 - \sum_{i=1}^{n} p^i \alpha_i \tag{7}$$

Substituting equation (6) and equation (7) into equation (5) and equating the terms with powers of p, we can obtain a series of linear equation.

The approximate for the solution and frequency are:

$$u = \lim_{p \to 1} \sum_{i=1}^{n} u_i \tag{8}$$

$$\omega^2 = 1 + \lim_{p \to 1} \sum_{i=1}^{n} \alpha_i \tag{9}$$

where α_i are arbitrary parameters that should be determined.

2.2. *Energy balance method*

In order to illustrating the basic concept of energy balance method, we consider nonlinear equation as follows:

$$\ddot{u} + f(u(t)) = 0 \tag{10}$$

Since u and t are generalized dimensionless displacement and time variables, respectively.

Its variational principle can be obtained:

$$\begin{aligned} J(u) &= \int_0^t \left(-\frac{\dot{u}^2}{2} + F(u) \right) dt \\ J(u) &= \int f(u) du \end{aligned} \tag{11}$$

Using $u\,(0) = A$,$\dot{u}\,(0) = 0$ as a boundary conditions, its Hamiltonian, therefore, can be written in the form:

$$H = \frac{\dot{u}^2}{2} + F(u) = F(A) \tag{12}$$

or:

$$R(t) = \frac{\dot{u}^2}{2} + F(u) - F(A) = 0 \tag{13}$$

Assume that its initial approximate guess can be expressed as:

$$u(t) = A\cos\omega t \tag{14}$$

Substituting Eq. (13) into u which term of (14) yields:

$$R(t) = \frac{\omega^2 A^2 \sin\omega t}{2} + F(A\cos\omega t) - F(A) = 0 \tag{15}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Since Eq. (16) is only an approximation to the exact solution, R cannot be made zero everywhere. Collocation at $\omega t = \pi/4$ gives:

$$\omega = \sqrt{\frac{2(F(A) - F(A\cos\omega t))}{A^2\sin^2\omega t}} = \sqrt{\frac{4(F(A) - F(\sqrt{2}A/2))}{A^2}} \tag{16}$$

3. Application

3.1. *Modified homotopy perturbation method:*

We can rewrite equation (1) as following form:

$$\ddot{u} + 1u - p\left[a_3u^3 + a_5u^5 + \cdots + a_{2q-1}u^{2q-1}\right] \qquad p \in [0,\, 1] \tag{17}$$

where p is embedding parameter, which is also used to expand u:

$$u = \sum_{i=1}^{n} p^i u_i \tag{18}$$

By expanding 1 as a coefficient of u, we can obtain:

$$\omega^2 = 1 + \sum_{i=1}^{n} p^i \alpha_i \tag{19}$$

Substitution equation (17) and equation (18) into equation (1) yields:

$$p^0 : \ddot{u}_0 + \omega^2 u_0 = 0 \tag{20}$$
$$p^1 : \ddot{u}_1 + \omega^2 u_1 = (\alpha_1)u_0 - a_3u^3 - a_5u^5 - \cdots - a_{2q-1}u^{2q-1} \tag{21}$$

By solving equation (19), we can obtain:

$$u_0(t) = A\cos\omega t \tag{22}$$

Substitution of equation (21) into the right side of equation (20) gives $\ddot{u}_1 + \omega^2 u_1 = \rho\,(\omega t)$, in which:

$$\rho\,(\omega t) = -\alpha_1 A \cos{(\omega t)} + a_3 A^3 \cos{(\omega t)}^3 + a_5 A^5 \cos{(\omega t)}^5 \\ + \cdots + a_{2q-1} A^{2q-1} \cos{(\omega t)}^{2q-1} \tag{23}$$

By using Fourier series, we can achieve secular term:

$$\rho(\omega t) = \sum_{n=0}^{\infty} \delta_{2n+1} \cos[(2n+1)\omega] \approx \delta_1 \cos(\omega t) \tag{24}$$

$$\delta_1 = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \rho(\varphi) \cos(\varphi) d\varphi = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \big(-\alpha_1 A \cos^2(\varphi) + a_3 A^3 \cos^4(\varphi) + \cdots \\ + a_{2q-1} A^{2q-1} \cos^{2q}(\varphi)\big)\, d\varphi \tag{25}$$

We know:

$$\int_0^{\frac{\pi}{2}} \cos^{2q}(t) = \frac{\pi}{2} \prod_{j=1}^{q} \left(\frac{2j-1}{2j} \right) \tag{26}$$

Using equation (23) yields:

$$\delta_1 = 2 \left\{ \sum_{i=2}^{2q-1} \left[a_{2i-1} A^{2i-1} \prod_{j=1}^{q} \left(\frac{2j-1}{2j} \right) \right] \right\} - \alpha_1 A \tag{27}$$

Avoiding secular term requires $\delta_1 = 0$, so:

$$\alpha_1 = 2 \left\{ \sum_{i=2}^{2q-1} \left[a_{2i-1} A^{2i-1} \prod_{j=1}^{q} \left(\frac{2j-1}{2j} \right) \right] \right\} \tag{28}$$

From equation (7) and by setting p = 1:

$$\omega^2 = 1 + \alpha_1 \tag{29}$$

Therefore we can obtain angular frequency:

$$\omega_{MHPM} = \sqrt{2 \left\{ \sum_{i=2}^{2q-1} \left[a_{2i-1} A^{2i-1} \prod_{j=1}^{i} \left(\frac{2j-1}{2j} \right) \right] \right\} + 1} \tag{30}$$

3.2. *Energy balance method*

We can rewrite equation (1) as following form:

$$\ddot{u} + f(u)u = 0 \tag{31}$$

where $f\,(u)$ is:

$$f(u) = 1 + a_3u^2 + a_3u^4 + \cdots + a_{2q-1}u^{2q-2} \tag{32}$$

Its variational principle can be easily obtained as:

$$J(x) = \int_0^t \left(-\frac{1}{2}\dot{u}^2 + \frac{u^2}{2} + \frac{a_3u^4}{4} + \frac{a_5u^6}{6} + \cdots + \frac{a_{2q-1}u^{2q-2}}{2q} \right) \tag{33}$$

The Hamiltonian of Eq. 33, can be yield in the form:

$$\begin{aligned} H &= \frac{\dot{u}}{2} + \frac{u^2}{2} + \frac{a_3u^4}{4} + \frac{a_5u^6}{6} + \cdots + \frac{a_{2q-1}u^{2q}}{2q} \\ &= \frac{A^2}{2} + \frac{a_3A^4}{4} + \frac{a_5A^6}{6} + \cdots + \frac{a_{2q-1}A^{2q}}{2q} \end{aligned} \tag{34}$$

or:

$$\begin{aligned} R(t) &= \frac{\dot{u}}{2} + \frac{u^2}{2} + \frac{a_3u^4}{4} + \frac{a_5u^6}{6} + \cdots + \frac{a_{2q-1}u^{2q}}{2q} \\ &-\frac{A^2}{2} - \frac{a_3A^4}{4} - \frac{a_5A^6}{6} - \cdots - \frac{a_{2q-1}A^{2q}}{2q} = 0 \end{aligned} \tag{35}$$

Oscillation systems contain two important physical parameters, i.e. the frequency ω and the amplitude of oscillation, A. Substituting $u(t) = A\cos(\omega t)$ as a trial function into (35) the following residual can be obtained:

$$\begin{aligned} R(t) &= \frac{1}{2}A^2\omega^2\sin^2\omega t + \frac{1}{2}A^2\cos^2\omega t + \frac{1}{4}a_3A^4\cos^4\omega t + \frac{1}{6}a_5A^6\cos^6\omega t + \cdots \\ &+\frac{1}{2q}a_{2q-1}A^{2q}\cos^{2q}\omega t - \left(\frac{A^2}{2} + \frac{a_3A^4}{4} + \frac{a_5A^6}{6} + \cdots + \frac{a_{2q-1}A^{2q}}{2q} \right) = 0 \end{aligned} \tag{36}$$

If we collocate $\omega t = \pi/4$, we obtain:

$$\omega_{EBM} = \sqrt{4\sum_{i=2}^{2q-1} \left\{ \frac{a_{2i-1}A^{2i-1}}{2i} \left[1 - \left(\frac{\sqrt{2}}{2} \right) \right]^{2i} \right\} + 1} \tag{37}$$

4. Numerical Cases

In this study, the objective is applying MHPM and EBM to obtain an explicit analytic solution of the generalized Duffing oscillation problem. In Table.1 the comparison between the period obtained from analytical approximate and exact solution [4] for a range of oscillation amplitudes have been presented and also the error analysis have been calculated.

According to error analysis in Tab. 1, the results which obtained from EBM have more accuracy with exact result in comparison with MHPM. Another consequence is achieved from these figures, is good adjustment between the exact solution and approximate results. High accuracy and validity (Figs. 1) reveals that both methods are powerful and effective to use Solution gives us possibility to obtain frequency in different cases with different values of power, so many of prior researches in Duffing nonlinear equations are special cases of this general solution.

Table 1 Comparison of the approximate periods with the exact period when A= 0.1, 1, 10

A	a_3	a_5	a_7	*Type of Equation*	T_{EXACT}	T_{MHPM} $Er(\%)$	T_{EBM} $Er(\%)$
0.1	1	0	0	$q = 2$ $\ddot{u} + u + a_3u^3 = 0$	6.2598	6.2596 0.002%	6.2596 0.002%
	1	1	0	$q = 3$ $\ddot{u} + u + a_3u^3 + a_5u^5 = 0$	6.2564	6.2562 0.001%	6.2563 0.001%
	1	1	1	$q = 4$ $\ddot{u} + u + a_3u^3 + a_5u^5 + a_7u^7 = 0$	6.2596	6.2596 0.000%	6.2596 0.000%
1	1	0	0	$q = 2$ $\ddot{u} + u + a_3u^3 = 0$	4.7682	4.7497 0.314%	4.7497 0.314%
	1	1	0	$q = 3$ $\ddot{u} + u + a_3u^3 + a_5u^5 = 0$	4.1218	4.0771 1.082%	4.1106 0.271%
	1	1	1	$q = 4$ $\ddot{u} + u + a_3u^3 + a_5u^5 + a_7u^7 = 0$	3.7504	3.6758 1.982%	3.7535 0.082%
10	1	0	0	$q = 2$ $\ddot{u} + u + a_3u^3 = 0$	0.7363	0.7207 2.116%	0.7207 2.116%
	1	1	0	$q = 3$ $\ddot{u} + u + a_3u^3 + a_5u^5 = 0$	0.0835	0.0789 5.501%	0.0816 2.277%
	1	1	1	$q = 4$ $\ddot{u} + u + a_3u^3 + a_5u^5 + a_7u^7 = 0$	0.0092	0.0084 8.696%	0.0091 1.081%

5. Conclusions

In this paper, the generalized Duffing equation is investigated with application of Modified homotopy perturbation method (MHPM) and Energy Balance Method (EBM) which are two powerful and efficient methods. At first the basic concepts of these methods are explained, application of solution procedure illustrated in both MHPM and EBM and finally the obtained results are compared with exact integration solution. The advantages of using these methods are high accuracy and simple procedure in comparison to exact solution. Comparison this general solution to exact integration results shows the high accuracy and validity of these methods even in high strength type of nonlinearity.

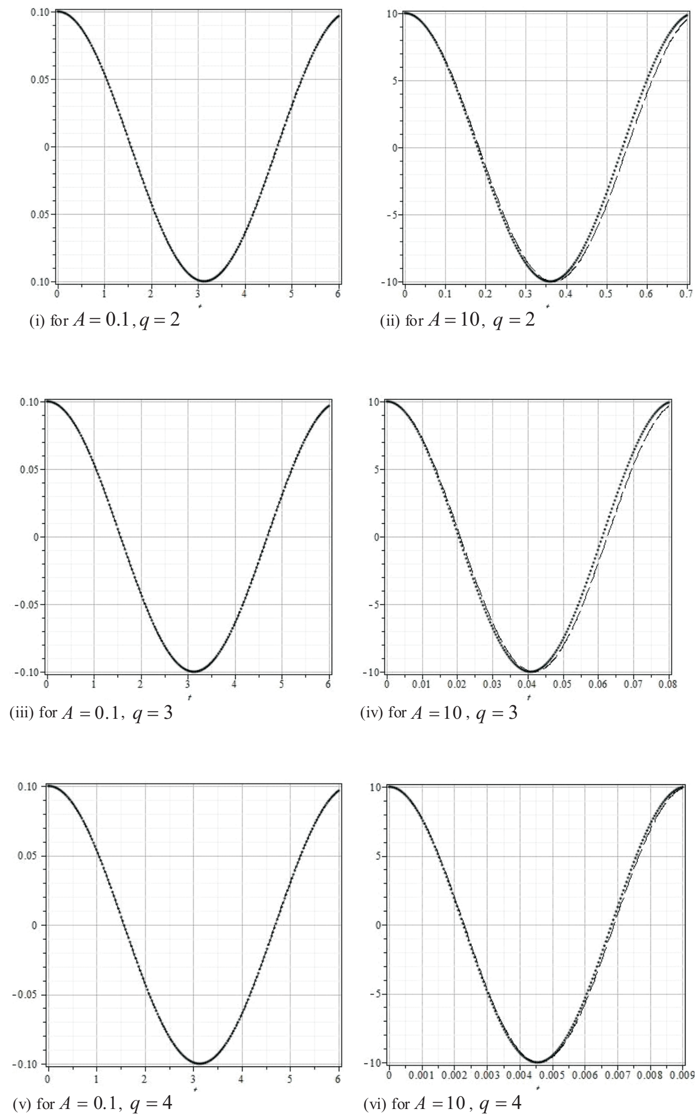

Figure 1 Comparison between EBM (Dash Line) with numerical ones (Dot) with different A

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