Mechanics and Mechanical Engineering
Vol. 21, No. 4 (2017) 11–16

Optimal Generalized Hohmann Transfer with Plane Change Using Lagrange Multipliers

Osman M. Kamel
Astronomy and Space Science Dept.,
Faculty of Sciences, Cairo University
Giza, Egypt
kamel_osman@yahoo.com

Adel S. Soliman
Mohamed R. Amin
Theoretical Physics Dept.
National Research Center
Dokki, Giza, Egypt
adelssoliman7@yahoo.com

Received (29 August 2016)
Revised (6 October 2016)
Accepted (11 November 2016)

The optimized orbit transfer of a space vehicle, revolving initially around the primary, in a similar orbit to that of the Earth around the Sun, in an elliptic trajectory, to another similar elliptic orbit of an adequate outer planet is studied in this paper. We assume the elements of the initial orbit to be that of the Earth, and the elements of the final orbit to be that of an outer adequate planet, Mars for instance. We consider the case of two impulse generalized Hohmann non coplanar orbits. We need noncoplanar (plane change) maneuvers mainly because: 1) a launch–site location restricts the initial orbit inclination for the vehicle; 2) the direction of the launch can influence the amount of velocity the booster must supply, so certain orientations may be more desirable; and 3) timing constraints may dictate a launch window that isn't the best, from which we must make changes[3]. We used the Lagrange multipliers method to get the optimum of the total minimum energy required ΔV_T , by optimizing the two plane change angles α_1 and α_2, where α_1 is the plane change at the first instantaneous impulse at peri–apse, and α_2 the plane change at the second instantaneous thrust at apo–apse. We adopt the case of Earth – Mars, as a numerical example.

Keywords: orbital mechanics, elliptic Hohmann transfer with plane change, optimization problem, Lagrange multipliers.

1. Introduction

Walter Hohmann (1925), proposed a theory which suggested the minimum change in velocity transfer could be achieved between coplanar circular and elliptic orbits by using two tangential burns (two impulse) [1]. As for the derivations of the velocity change requirements ΔV_1, ΔV_2 and transfer time, we can draw a graph which illustrates total energy/satellite mass as a function of orbit period $P = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$ that means a plot of $\frac{-\mu}{2a}$ versus $\left(\frac{2\pi}{\sqrt{\mu}}\right) a^{3/2}$ Plotted results are extensively established [1], [2]. For classical Hohmann transfer if $\frac{r_2}{r_1}\langle 15.58 r_2 \rangle r_1$ is not satisfied, then the Hohmann transfer is no longer optimal. For these conditions Bi–elliptic transfers are always more economical in propellant than Hohmann transfer configurations [1], [3]. The Hohmann transfer is a relatively simple maneuver, especially the classical model. It may be simplified or complicated easily, and we may encounter very difficult situations. This can be easily seen from the literature of orbit transfer [4]. There exist four feasible Hohmann configurations according to the coincidence of peri-apse and apo–apse of the three ellipses. We consider the first of them [5],[6]. Radius or major axis change, in the process of orbit transfer may be coupled by a plane change for the circular or elliptic orbit transfer. This is an important practical procedure. The optimal two impulse transfer that satisfy these conditions is the Hohmann transfer, with split plane change. The first ΔV_1 thrust not only produces a transfer ellipse but also induce a rotation of the orbital plane. At the second impulse, a second tilt is induced as well as the production of the final elliptic orbit. An engine firing in the out-of plane direction is required for the change of plane. The point of firing becomes a point in the new orbit, and the burn point becomes the intersection of the current orbit and the desired orbit. Definitely, we should perform plane change in the smartest way, since it is fuel expensive, anyway you do them. Even without the examination of the specific equations, planning a space mission, reduces to a problem of geometry, timing, mechanics of orbital motion, and a lot of common sense.

2. Method and results

In this article we investigate the generalized Hohmann orbit transfer with split–plane change. We take into account, the first configuration, where the apo–apse of the transfer orbit coincides with the apo–apse of the final orbit, and the peri–apse of the initial and the transfer orbit are coincident], Fig. 1.
ΔV_1 produces at peri–apse of initial orbit, a transfer ellipse as well as a plane change α_1. Similarly at apo-apse, ΔV_2 rotates the orbit plane through an angle $\alpha_2 = \theta - \alpha_1$, and designs the final elliptic orbit as shown in Fig. 1.
From Fig. 1, we notice that:

$$\begin{aligned} r_1 &= a_1 (1 - e_1) = a_T (1 - e_T) \\ r_2 &= a_2 (1 + e_2) = a_T (1 + e_T) \end{aligned} \tag{1}$$

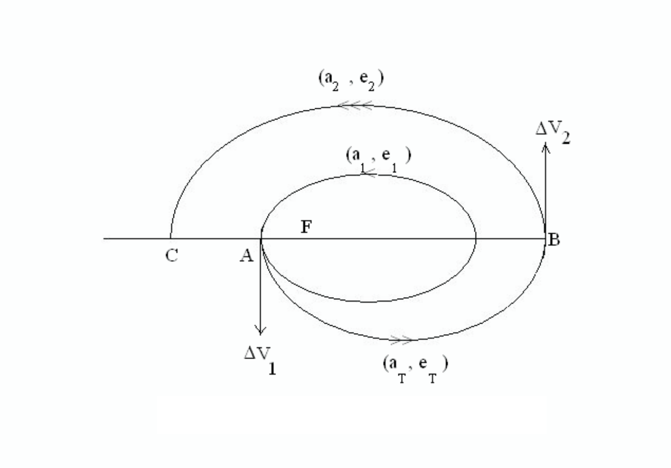

Figure 1

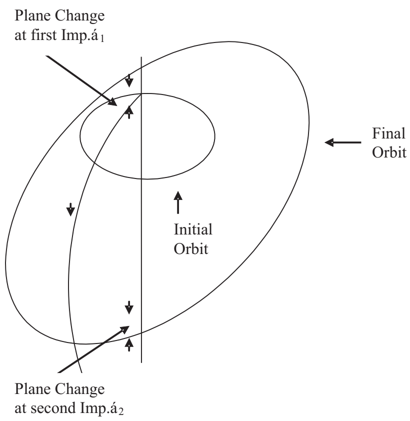

Figure 2

where:

$$V_{P_i} = \sqrt{\frac{\mu\,(1+e_1)}{r_1}} \qquad V_{P_{tr}} = \sqrt{\frac{\mu\,(1+e_T)}{r_1}}$$
$$V_{A_{tr}} = \sqrt{\frac{\mu\,(1-e_T)}{r_2}} \qquad V_{A_f} = \sqrt{\frac{\mu\,(1-e_2)}{r_2}} \tag{2}$$

The increments of velocities at peri–apse and apo–apse of the elliptic transfer orbit is given by [6]:

$$\Delta V_1^2 = \frac{\mu}{r_1}\left(1+e_1\right)+\left(1+e_T\right)-\frac{2\mu}{r_1}\sqrt{\left(1+e_1\right)\left(1+e_T\right)}\cos\alpha_1 \tag{3}$$

$$\Delta V_2^2 = \frac{\mu}{r_2}\left(1-e_2\right)+\left(1-e_T\right)-\frac{2\mu}{r_2}\sqrt{\left(1-e_2\right)\left(1-e_T\right)}\cos\alpha_2 \tag{4}$$

Let:

$$A = \frac{\mu}{r_1}\left(1+e_1\right) \qquad B = \frac{\mu}{r_1}\left(1+e_T\right)$$

$$C = \frac{\mu}{r_2}\left(1-e_2\right) \qquad D = \frac{\mu}{r_2}\left(1-e_T\right)$$

$$\begin{aligned} \Delta V_T &= \Delta V_1 + \Delta V_2 \\ &= \left(A+B-2\sqrt{AB}\cos\alpha_1\right)^{1/2}+\left(C+D-2\sqrt{CD}\cos\alpha_2\right)^{1/2} \end{aligned} \tag{5}$$

Let θ is arbitrary and given by $\theta = \alpha_1 + \alpha_2$, and e_T can be calculated from the formulae:

$$e_T = \frac{r_2 - r_1}{r_2 + r_1}$$

From the relation, [7]

$$\frac{r_2}{r_1} = \frac{\sin\theta/2}{1-2\sin\theta/2} \qquad r_2 > r_1 \tag{6}$$

we get the value of θ. Since r_1 and r_2 are known.

3. Lagrange multipliers method

We seek to minimize a function $\Delta V_T(\alpha_1, \alpha_2$ subject to $\theta = \alpha_1 + \alpha_2$
The Lagrange function F is constructed as [8]:

$$F\left(\alpha_1, \alpha_2, \lambda\right) = \Delta V_T(\alpha_1, \alpha_2) - \lambda\theta\left(\alpha_1, \alpha_2\right) \tag{7}$$

where λ is Lagrange multipliers.
The extreme points of the ΔV_T and the Lagrange multipliers λ satisfy:

$$\nabla F = 0 \quad i.e. \quad \frac{\partial F}{\partial \alpha_1} = 0 \quad \frac{\partial F}{\partial \alpha_2} = 0 \quad \frac{\partial F}{\partial \lambda} = 0 \tag{8}$$

Then, we have three equations in three unknowns, therefore we can get the values of $(\alpha_1)_{opt.}$, $(\alpha_2)_{opt.}$, and then get $(\Delta V_T)_{opt.}$

4. Numerical results

For Earth – Mars subsystem, we have [9]:
a_1 = semi – major axis of Earth = 1 AU.
a_2 = semi – major axis of Mars = 1.5237 AU.
e_1 = eccentricity of Earth = 0.0167.
e_2 = eccentricity of Mars = 0.0934.
Using the relation (6), we find that $\theta = 45.42$ deg, from Eq. (7), and applying (8), we find $(\alpha_1)_{opt.} = 1.22$ deg, and $(\alpha_2)_{opt.} = 44.20$ deg, Substitution in (5), we get $(\Delta V_T)_{opt.} = 15$ m/sec.

5. Discussion

The Hohmann transfer is an optimal two impulse transfer. We suppose that the first increment at peri-apse ΔV_1, not only produces a transfer elliptic orbit, but also rotates the orbital plane by an optimal angle α_1.
At apo–apse the second increment of velocity ΔV_2 will produce the trajectory of the final elliptic orbit and rotates the orbit plane by an angle $\alpha_2 = \theta - \alpha_1$. We have $\Delta V_T = \Delta V_1 + \Delta V2$. For the minimization of ΔV_T we apply Lagrange multipliers method to obtain the value of the optimized α_1 i.e. $(\alpha_1)_{opt.}$, whence $(\alpha_2)_{opt} = \theta - (\alpha_1)_{opt.}$. By substitution of $(\alpha_1)_{opt.}$, and $(\alpha_2)_{opt}$ we can easily evaluate $(\Delta V_T)_{Min}$ from Eq. (5) we notice that the first angle α_1 is less than the second angle α_2.
In this article we consider as a numerical example, the case of Earth – Mars subsystem.

References

[1] **Prussing, J. E. and Conway, B. A.**: Orbital Mechanics, *Oxford University Press*, **1993**.

[2] **Ehricke, K. A.**: Space Flight, *Dynamics, Van Nostrand*, Vol. 2, **1962**.

[3] **Vallado, D. A.**: Fundamentals of Astrodynamics and Applications, 3^{rd} edition, *Springer*, **2007**.

[4] **Lawden, D. F.**: Optimal Transfers Between Coplanar Elliptical Orbits, *J. Guidance, Engineering notes*, 15, 3, **1991**.

[5] **Kamel, O. M. and Soliman, A. S.**: On transfer between elliptical coplanar coaxial orbits, *Bulletin of the Faculty of Sciences*, Cairo University, 67, **1999**.

[6] **Kamel, O. M. and Soliman, A. S.**: On the generalized Hohmann transfer with plane change using energy concepts, Part I, *Mechanics and Mechanical Engineering*, 15, 2, 183–191, **2011**.

[7] **Chobotov, V. A.**: Orbital Mechanics, 3^{rd} edition, *AIAA, Education Series*, **2002**.

[8] **Bertsekas, D. P.**: Constrained Optimization and Lagrange Multiplier Methods *Academic Press*, 1 st ed., **1982**.

[9] **Murray, C. D. and Dermott, S. F.**: Solar System Dynamics, *Cambridge Universiy Press*, **1999**.

Nomenclature

ΔV_1 – increment of velocity at peri–apse impulse.
ΔV_2 – increment of velocity at apo–apse impulse.
$\Delta V_T = \Delta V_1 + \Delta V_2$.

μ – constant of gravitation.
a_1 – semi–major axis of initial orbit.
a_2 – semi–major axis of final orbit.
a_T – semi–major axis of transfer orbit.
e_1 – eccentricity of initial orbit.
e_2 – eccentricity of final orbit.
e_T – eccentricity of transfer orbit.
r_1 – initial radius (classical Hohmann).
r_2 – final radius (classical Hohmann).
α_1 – plane change at peri-apse.
α_2 – plane change at apo-apse.
$\theta = \alpha_1 + \alpha_2$ – total plane change.