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Time-Delay Two-Dimension Mathieu Equation in Synchrotron Dynamics

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Two dimensions Mathieu equation containing periodic terms as well as the delayed parameters has been investigated in the present work. The present system represents to a generalized form of the one-dimension delay Mathieu equation. The mathematical difficulty for delay the coupled Mathieu equation has been overcome by using the matrices method. Properties of inverse complex matrices enable us to transform the vector form of the solvability conditions to the scalar form. Small oscillation about a marginal state is introduced by using the method of multiple scales. Stability criteria for the complex matrices have been established and lead to obtain resonance curves. The analysis has been extended so that the delay 2-dimensions Mathieu equation containing weak complex damping part. Stability conditions and the transition curves that included the influence of both the delayed as well the complex damping terms has been obtained. The transition curves are analyzed using the method of harmonic balance. We note that the delayed higher dimension of the parametric excitation has a great interest and application to the design of nuclear accelerators.

Keywords: Synchrotron dynamics, time-delayed, two-dimension complex damped Mathieu equation, stability analysis, parametric resonance, inverse complex matrices method.

1. Introduction

There are a number of analytical solutions of Mathieu equation remains to be very actually for its applications in physical sciences and in engineering. On the other hand, an analytical solution of Mathieu equation has also the mathematical theoretical aspect. It is determined by the fact that the solution of a number of differential equations is reduced to the solution of Mathieu equation [1, 2].

There are a lot research in literature has made to understanding of the one-dimensional dynamic systems which are described by one dimension Mathieu equation. There are many literatures deal with the analysis and applications of the Mathieu equation [1-5]. Choudhury and Guha [6], have restricted their attention to a particular class of damped Mathieu equation, where the damping coefficient is a function on time. In their analysis, due to the specific nature of time dependence, they employed the technique that first introduced by Bartucelli and Gentile [7]. This technique enables them to construct a first-integral and consequently, the solution of the resulted equation in a manner very close to that of linear harmonic oscillator. Recently, Turovtsev et al. [8] made an algorithm for calculating the spectrum and the wave functions of stationary states for one-dimensional rotation in the basis set of the Mathieu functions.

Recently attentions to dynamical systems having higher dimensions are increased. The outstanding challenge for future research is the analysis of higher-dimensional dynamical systems. Such systems are known to exhibit regular and chaotic behavior at various levels, but their solutions are, of course, more difficult to analyze globally. An important task in elucidating the properties of such dynamical systems is finding some of their fundamental stable and unstable periodic solutions, and studying the motion in their vicinity. Mohamed et al [9, 10] studied a fluid layer subjected to a periodic force which controlled by two-dimension Mathieu equation having real coefficients. Classical and quantum modes of coupled Mathieu equations have been addressed by H. Landa et al [11]. Due to the presence of viscous flow with non-zero streaming, the coefficients of Mathieu equations becomes of a complex form El-Dib, Y. O. and Ghaly, A. Y. [12] studied Kelvin–Helmholtz waves propagating between two magnetic fluids and deriving the Mathieu equation governing the interfacial displacement and having complex coefficients. They found stability behavior near marginal state response.

Two dimension Mathieu equations having complex coefficients has been addressed by Y. O. El-Dib and R. T. Matoog [13, 14] and Y. O. El-Dib [15] in the area of electro-hydro-dynamic stability of double interfaces, when they are influenced by periodic electric field or an oscillatory stationary streaming flow. These authors discussed some special cases in order to avoid the complexity for this system. They found stability behavior near marginal state response. S. A. Alkharashi [16] has deriving two simultaneous Mathieu equations of damping terms having complex coefficients in the stability of three layers of immiscible liquids. The fluids are subjected to a uniform horizontal electric field and periodic velocities. Recently, in (2016) A. R. AlHamdan and S. A. Alkharashi [17] have discussed the instability of three horizontal finite layers of immiscible fluids in porous media subjected to a horizontal magnetic field. The problem concerned with a periodic velocity. They derived two simultaneous Mathieu equations of damping terms having complex coefficients.

Stability of damped Mathieu's equation with time-periodic coefficients and time-delayed Mathieu's equation has been considered by T. Insperger and G. Stépán [18] and by N. K. Garg et al [19]. Z. Ahsan et al [20] considered damped Mathieu equation with two different points delayed. The dynamics of a type of particle accelerator called a synchrotron, in which particles are made to move in nearly circular orbits of large radius. The stability of the transverse motion of such a rotating particle may be modeled as being governed by Mathieu's equation. For a train of two such particles the equations of motion are coupled due to plasma interactions and resistive wall coupling effects [21]. A. Bernstein & R.H. Rand [22] has address investigation of coupled Parametrically Driven Modes in Synchrotron Dynamics.

They studied a system consisting of a train of two particles which is modeled as two coupled nonlinear Mathieu equations with delay coupling. Recently, in (2016) the delay-coupled Mathieu equations in synchrotron dynamics have been addressed by A. Bernstein and R.H. Rand [23]. They investigate the dynamics of the model having two delay-coupled Mathieu equations. They interested in the form of the above equations comes from an application in the design of nuclear accelerators. They used the two-time scales method [15] to study the dynamics for their coupled Mathieu equations
In present work, we consider a generalized form of the delay-two coupled Mathieu equations considered by A. Bernstein and R.H. Rand [23]. In order to get a wide applications and for more generalization, the mathematical model has been extended to included complex damping coefficients.

2. Mathematical problem

The underlying mathematical problem of 2-dimension Mathieu equation, with a single point delay is given below:

$$\begin{aligned}&\frac{d^2x_k\left(t\right)}{dt^2}+\left(a_k+4\varepsilon q_k\cos^2\Omega t\right)x_k\left(t\right)+\left(a_j+4\varepsilon q_j=\cos^2\Omega t\right)x_j\left(t\right)\\&\varepsilon h_k x_k\left(t-\tau\right)+\varepsilon h_j x_j\left(t-\tau\right)\qquad k\neq j\qquad k,j=1,2\end{aligned}\tag{1}$$

where ε is a non-zero small parameter, Ω is a frequency of the external excitation, $q's$ are the amplitude of external excitation, t is an independent parameter, $a's$ are real constants and h_k scales the influence of delay state. Equation (1) represents 2-coupled delayed Mathieu's equation, which can be introduced into the vector extension of a standard Mathieu equation as shown below:

$$\frac{d^2}{dt^2}\underline{X}\left(t\right)+\left(\underline{A}+4\varepsilon\underline{Q}\cos^2\Omega t\right)\underline{X}\left(t\right)=\varepsilon\underline{HX}\left(t-\tau\right)\tag{2}$$

where $\underline{A}$, $\underline{Q}$ and $\underline{H}$ are non-singular square matrices of 2×2 type while the vector $\underline{X}(t)=\left(x_1(t),\ x_2(t)\right)^T$ is of 2×1 type and the upper T refer to the transposed of matrix. This system has a periodic solution, in which, the unperturbed form for the system (2) has the form:

$$\frac{d^2}{dt^2}\underline{X}\left(t\right)+\underline{AX}\left(t\right)=0\tag{3}$$

From the analysis of matrices, solutions for the vector equations (3) may be written in the following form:

$$\underline{X}(t)=\underline{R}_j\left(\pi_j e^{i\omega_j t}+\pi_j^* e^{-i\omega_j t}\right)\qquad j=1,2\tag{4}$$

where $\underline{R}_j$ is a constant vector of 2×1 type, π_j, $j=1,2$ is a constant of integration and π_j^* is the complex conjugate of the constantπ_j. The natural frequency ω_j is the eigenvalues that given by the following characteristic equation:

$$\omega^4-tr\left(\underline{A}\right)\omega^2+\det\left(\underline{A}\right)=0\tag{5}$$

These eigenvalues are:

$$\omega_{1,2}^2 = \tfrac{1}{2} tr\left(\underline{A}\right) \pm \tfrac{1}{2}\sqrt{tr^2\left(\underline{A}\right) - 4\det\left(\underline{A}\right)}. \tag{6}$$

The necessary and sufficient condition for stability, at this case, is that ω_j^2 must be real and positive. It is easily verified from (5) that the restriction of the stability for the unperturbed equation (3) implies the following conditions:

$$tr\left(\underline{A}\right) > 0 \quad \det\left(\underline{A}\right) > 0 \quad \text{and} \quad tr^2\left(\underline{A}\right) - 4\det\left(\underline{A}\right) > 0 \tag{7}$$

In what follows we try to find an approximate solution for equation (2) about the periodic solution given by (4). We shall dealing with producing approximate solution for equation (2) in the non-vanishing of the small parameter ε.

3. Stability analysis when the time-depended in Mathieu equations is switch on

We shall discuss the stability of (2) using asymptotic expansion treatment. We shall apply the well-known of the multiple scale method [3]. This method enables us to discuss the stability of the problem (2).

On applying the method of multiple scales we may use the scale T_0, T_1 such that $T_n = \varepsilon^n t$, $n = 0, 1, 2, ...$ T_0 is the variable appropriate to fast variable, and T_1 is the slow variable. In addition, the delay time can scaled as $\tau_n = \varepsilon^n \tau$. The differential operator can now expressed as the partial derivative expansions:

$$D \equiv D_0^2.. + 2\varepsilon D_0 D_1.. + ..., \qquad D_n.. \equiv \frac{\partial ..}{\partial T_n} \tag{8}$$

Assuming that ω_1^2 and ω_2^2 are both real and positive, then the dependent vector variable $\underline{X}(t)$ can be expanded in the form:

$$\underline{X}(t,\varepsilon) = \underline{X}_0(T_0,T_1) + \varepsilon \underline{X}_1(T_0,T_1) + ... \tag{9}$$

where the vector $\underline{X}_0(T_0,T_1)$ has been found to be:

$$\underline{X}(T_0,T_1) = \underline{R}_j\left(\pi_j(T_1)e^{i\omega_j T_0} + \pi_j^*(T_1)e^{-i\omega_j T_0}\right) \tag{10}$$

While the perturbed vector $\underline{X}_1(T_0,T_1)$ is given by:

$$\begin{aligned}
&\left(D_0^2 \underline{I} + \underline{A}\right)\underline{X}_1(T_0,T_1) = \\
&-\left[\left(2i\omega_j + \varepsilon\tau\underline{H}e^{-i\omega_j\tau}\right)D_1 - \underline{H}e^{-i\omega_j\tau} + 2Q\right]\underline{R}_j\pi(T_1)e^{i\omega_j T_0} \\
&-\underline{Q}\left(e^{i(\omega_j+2\Omega)T_0} + e^{i(\omega_j-2\Omega)T_0}\right)\underline{R}_j\pi_j(T_1) \\
&\left[\left(\varepsilon\tau\underline{H}e^{i\omega_j\tau} - 2i\omega_j\right)D_1 - \underline{H}e^{i\omega_j\tau} + 2\underline{Q}\right]\ \underline{R}_j\pi_j^*(T_1)e^{-i\omega_j T_0} \\
&-\underline{Q}\left(e^{-i(\omega_j-2\Omega)T_0} + e^{-i(\omega_j+2\Omega)T_0}\right)\underline{R}_j\pi_j^*(T_1)
\end{aligned} \tag{11}$$

where the unknown function $\pi_j\left(T_1 - \varepsilon\tau\right)$ is expanded about the un-delayed variable and used. The most general way to express it in terms of T_1 is with a Taylor series:

$$\pi_j\left(T_1 - \varepsilon\tau\right) = \pi_j\left(T_1\right) - \varepsilon\tau\, D_1\pi_j\left(T_1\right) + ... \tag{12}$$

The above equation contains secular vectors that are proportional to the factor $(e^{\pm i\omega_j T_0})$. Before eliminating these secular vectors we are in need to distinguish between several possible combinations of Ω, ω_1 and ω_2. These cases of Ω near $\omega_j$or near the combinations and $\frac{1}{2}(\omega_1 \pm \omega_2)$ are known as resonance cases. The non-resonant case arises when Ω away from ω_1, ω_2 and $\frac{1}{2}(\omega_1 \pm \omega_2)$.

3.1. *The non-resonance case*

The elimination of secular terms, in equation (11), at the non-resonant case leads to:

$$\left(\varepsilon\tau \underline{H} e^{-i\omega_j\tau} + 2i\omega_j \underline{I}\right) \underline{R}_j D_1 \pi + \left(-\underline{H} e^{-i\omega_j\tau} + 2\underline{Q}\right) \underline{R}_j \pi = 0 \tag{13}$$

with its complex conjugate form. Equation (13) is known as the amplitude equation. In order to transform the vector equation to a scalar case, one can multiply both sides of (13) from the left by $\underline{R}_j^T \left(2i\omega_j \underline{I} + \varepsilon\tau \underline{H} e^{-i\omega_j\tau}\right)^{-1}$. The use of the following normalized condition is useful:

$$\frac{\underline{R}_j^T \underline{R}_j}{\left|\underline{R}_j^T\right| \left|\underline{R}_j\right|} = 1 \tag{14}$$

Consequently, equation (13) transformed to the following scalar equation:

$$D_1 \pi_j(T_1) + (p_j + ik_j)\, \pi_j(T_1) = 0 \tag{15}$$

where:

$$\underline{S}_j = \frac{\underline{R}_j^T}{\underline{R}_j^T \underline{R}_j} \left(4\omega_j^2 \underline{I} + \varepsilon^2\tau^2 \underline{H}^2 - 4\varepsilon\tau\omega_j \underline{H} \sin\omega_j\tau\right)^{-1} \tag{16}$$

Equation (15) has been derived by help of the properties for the inverse of a non-singular square complex Matrix [24], in which,

$$\begin{aligned} &\left(\pm 2i\omega_j \underline{I} + \varepsilon\tau \underline{H} e^{\mp i\omega_j\tau}\right)^{-1} = \left(4\omega_j^2 \underline{I} + \varepsilon^2\tau^2 \underline{H^2} - 4\varepsilon\tau\omega_j \underline{H} \sin\omega_j\tau\right)^{-1} \\ &\left[\varepsilon\tau \underline{H} \cos\omega_j\tau + i\left(\mp 2\omega_j \underline{I} \pm \varepsilon\tau \underline{H} \sin\omega_j\tau\right)\right] \end{aligned} \tag{17}$$

The amplitude equation (15) is a single first–order differential equation with complex coefficients. It is having the following exponential solution:

$$\pi_j(T_1) = \Lambda_j e^{-(p_j + ik_j)T_1} \tag{18}$$

where Λ_j is an arbitrary complex constant and

$$p_j = 2\varepsilon\tau\left(\underline{S}_j \underline{H} \underline{Q} \underline{R}_j\right) \cos\omega_j\tau - \varepsilon\tau\left(\underline{S}_j \underline{H}^2 \underline{R}_j\right) + 2\omega_j\left(\underline{S}_j \underline{H} \underline{R}_j\right) \sin\omega_j\tau \tag{19}$$

$$k_j^{\pm} = \pm 2\varepsilon\tau\left(\underline{S}_j \underline{H} \underline{Q} \underline{R}_j\right) \sin\omega_j\tau \mp 4\omega_j\left(\underline{S}_j \underline{Q} \underline{R}_j\right) \pm 2\omega_j\left(\underline{S}_j \underline{H} \underline{R}_j\right) \cos\omega_j\tau \tag{20}$$

This solution is in the form of the growth rate. Therefore, the Routh-Hurwitz criteria [25] enable us to judge the stability of it. According to these criteria, the stability of the problem in the non-resonant case depends mainly on the negative real part for the value given in the exponent of (18). Thus the stability constrain, in the non-resonance case, is found as the scalar quantity p_j has positive value. In the limiting case for un- delayed two-dimension Mathieu equation, the above condition will be identically to zero and hence the solution (18) becomes:

$$\pi_j(T_1) = \Lambda_j \exp.\left(-i\frac{\underline{R}_j^T \underline{Q}\underline{R}_j}{2\omega_j \left|\underline{R}_j^T\right| \left|\underline{R}_j\right|}\right) T_1 \tag{21}$$

This shows that the marginal stability reveals at the non-resonance case and the presence of the time-delay term plays a damping role in the stability picture under a certain condition:

$$2\varepsilon\tau\left(\underline{S}_j\underline{H}\underline{Q}\underline{R}_j\right)\cos\omega_j\tau - \varepsilon\tau\left(\underline{S}_j\underline{H}^2\underline{R}_j\right) + 2\omega_j\left(\underline{S}_j\underline{H}\underline{R}_j\right)\sin\omega_j\tau > 0. \tag{22}$$

3.2. *The resonant case of* Ω *near* ω_j

The selection of a particular excitation frequency Ω is anticipated mathematically by introducing a detuning parameter σ in equation (11) to convert the small divisor term into secular term as follows:

$$\Omega = \omega_j + \varepsilon\sigma \tag{23}$$

and write:

$$-i(\omega_j - 2\Omega)T_0 = i\omega_j T_0 + 2i\sigma T_1 \tag{24}$$

Using (23) the small-divisor term arising from $\exp\left[\pm i\left(\omega - 2\Omega\right)T_0\right]$ in equation (11) can be transformed into a secular term. Then, remove the source of secular terms. At this stage, the following solvability condition is imposed:

$$D_1\pi_j(T_1) + \left(p_j + ik_j^+\right)\pi_j(T_1) + \left(\tilde{p}_j + i\tilde{k}_j\right)\pi_j^*(T_1)e^{2i\sigma T_1} = 0 \tag{25}$$

with its complex conjugate form. This is the scalar amplitude equations governed the stability behavior at the resonance case. This deferential equation with complex coefficients and having parametric term associated with the complex conjugate of the variable $\pi_j(T_1)$. Solution of equation (25) can be sought in the form:

$$\pi_j\left(T_1\right) = \lambda_j e^{(\Theta + i\sigma)T_1} \tag{26}$$

where λ_j is a complex constant and Θ is a growth rate. Equation (25) with its complex conjugate form represents a system of first-order differential equations and having the characteristic matrix:

$$\left(\underline{P} + i\underline{K} + \Theta\underline{I}\right)\underline{X}(t) = 0 \tag{27}$$

The eigenvalues for the characteristic exponent is found from the following characteristic equation:

$$\begin{aligned}&\det\left(\underline{P} + i\underline{K} + \Theta\underline{I}\right) = \Theta^2 + \left[tr(\underline{P}) + i\,tr\left(\underline{K}\right)\right]\Theta + \left[\det\left(\underline{P}\right) - \det\left(\underline{K}\right)\right]\\ &+ i\,tr\left(\underline{PK}^{-1}\right)\det\left(\underline{K}\right) = 0\end{aligned} \tag{28}$$

where the characteristic complex matrix $\left(\underline{P} + i\underline{K} + \Theta\underline{I}\right)$ having the following parts:

$$\underline{P} = \begin{pmatrix} p_j & \tilde{p}_j \\ \tilde{p}_j & p_j \end{pmatrix} \quad \text{and} \quad \underline{K} = \begin{pmatrix} \left(k_j^+ + \sigma\right) & \tilde{k}_j \\ -\tilde{k}_j & \left(k_j^- - \sigma\right) \end{pmatrix}$$

where $\tilde{p}_j$ and $\tilde{k}_j$ are given in the form:

$$\tilde{p}_j = \left(\underline{S}_j \underline{H} \underline{Q} \underline{R}_j\right) \varepsilon\tau \cos \omega_j \tau \tag{29}$$

$$\tilde{k}_j = \left(\underline{S}_j \underline{H} \underline{Q} \underline{R}_j\right) \varepsilon\tau \sin \omega_j \tau - 2\omega_j \left(\underline{S}_j \underline{Q} \underline{R}_j\right) \tag{30}$$

The construction of the matrix $\underline{P}$ is due to the presence of the delay coefficients as well as the time-delay parameter τ. The absence of the delay influence make this damping matrix coefficients $\underline{P}$ tends to zero, consequently, the characteristic equation (28) controls the marginal state for the stability picture as:

$$\det\left(\underline{K}_0 - \varpi \underline{I}\right) = \varpi^2 - \, tr\left(\underline{K}_0\right) \varpi + \det\left(\underline{K}_0\right) = 0, \tag{31}$$

where $\underline{K}_0$is the limiting case in the absence of the delay coefficients and it follows that the characteristic exponential should losing its real part to become $\Theta = i\varpi$. Accordingly, the solutions (26) will transformed to a periodic solutions and the marginal stability constrain at the present resonance case is that:

$$tr\left(\underline{K}_0\right) > 0 \quad \text{and} \quad \det\left(\underline{K}_0\right) > 0 \tag{32}$$

In the general case, where damping matrix coefficients $\underline{P}$ is presented, the characteristic equation (28) has a quadratic in Θ. It has two different complex roots Θ_1 and Θ_2. In ordinary differential equation with complex coefficients, the trivial solution is asymptotically stable if and only if all roots of the corresponding characteristic equation have negative real parts. Since the characteristic function is a polynomial, the well-known Routh-Hurwitz criterion [25] can be used in order to determine the negativity of the real parts of the roots Θ_1 and Θ_2 for characteristic equation (28). A necessary and sufficient condition for the stability of square matrices with complex entries is performed by Michael Y. Li and Liancheng Wang [26]. Thus, if all the eigenvalues, of the above characteristic equation, have negative real parts then the stability arises whence:

$$\begin{aligned} & tr\left(\underline{P}\right) > 0 \\ & \text{and} \\ & tr\left(\underline{P}\right)\left[tr\left(\underline{P}\right)\det\left(\underline{P}\right) - tr\left(\underline{P}\right)\det\left(\underline{K}\right) + tr\left(\underline{K}\right) tr\left(\underline{P}\underline{K}^{-1}\right)\det\left(\underline{K}\right)\right] \\ & - \left[tr\left(\underline{P}\underline{K}^{-1}\right)\det\left(\underline{K}\right)\right]^2 > 0 \end{aligned} \tag{33}$$

The first condition in (33) leads to $p_j > 0$, which equivalent to the stability condition at the non-resonance case. Therefore, the critical condition for stability is the second one in (33). First condition of the above stability criteria (33) depends on the influence of the amplitude of the parametric force $\underline{Q}$ and the amplitude of the delay terms $\underline{H}$as well as the delayed parameter τ. The influence of the detuning parameter σ has been included, only, in the second condition. This condition can be arranged in powers of the detuning parameter σ as:

$$\sigma^2 + 2\left(k_j^+ - k_j^-\right)\sigma + \left[2p_j^2 + \tfrac{1}{2}\left(k_j^+ - k_j^-\right)^2 - 2\left(\tilde{k}_j^2 + \tilde{p}_j^2\right)\right] > 0 \tag{34}$$

This condition has two zeros, namely,σ_1 and σ_2. Thus, the instability is found at the resonance case whence the detuning parameter σ lies inside the open interval(σ_2, σ_1). In terms of the frequency Ω, the transition curves separating stable region from unstable one corresponding to:

$$\begin{aligned} \Omega &= \omega_j - \varepsilon\left(k_j^+ - k_j^-\right) + \varepsilon\sigma_1 + ..., \\ \Omega &= \omega_j - \varepsilon\left(k_j^+ - k_j^-\right) - \varepsilon\sigma_2 + ... \end{aligned} \tag{35}$$

The region lies between the two curves represent the unstable (resonance) case, which impeded through the stable (the non-resonant) case.

3.3. *The resonant case of* Ω *near* $\frac{1}{2}(\omega_1 \pm \omega_2)$

Through this item, we shall consider the positive sign of ω_2. Meanwhile, the negative one may be obtained for replacing this sign in the results. We express the nearness of $\Omega$to$\frac{1}{2}(\omega_1 + \omega_2)$ by introducing the detuning parameter δ such that:

$$\Omega = \tfrac{1}{2}(\omega_1 + \omega_2) + \varepsilon\delta \tag{36}$$

Accordingly, we have:

$$\begin{aligned} -i\,(\omega_2 - 2\Omega)T_0 &= i\omega_1 T_0 + 2i\delta T_1 \\ -i\,(\omega_1 - 2\Omega)T_0 &= i\omega_2 T_0 + 2i\delta T_1 \end{aligned} \tag{37}$$

At this end, the secular terms appear in equation (11) can be rearranged to introducing the following two solvability conditions:

$$D_1\pi_1(T_1) + \left(p_1 + ik_1^+\right)\pi_1(T_1) + \left(\tilde{p}_{12} + i\tilde{k}_{12}\right)\pi_2^*(T_1)e^{2i\delta T_1} = 0 \tag{38}$$

$$D_1\pi_2^*(T_1) + \left(p_2 + ik_2^-\right)\pi_2^*(T_1) + \left(\tilde{p}_{21} - i\tilde{k}_{21}\right)\pi_1(T_1)e^{-2i\delta T_1} = 0 \tag{39}$$

They are not complex conjugate for each other. The scalar coefficients $\tilde{p}_{12}$, $\tilde{k}_{12}$, $\tilde{p}_{21}$ and $\tilde{k}_{21}$ are:

$$\begin{aligned} \tilde{p}_{12} &= \left(\underline{S}_1\underline{H}\underline{Q}\underline{R}_2\right)\varepsilon\tau\cos\omega_1\tau \\ \tilde{k}_{12} &= \left(\underline{S}_1\underline{H}\underline{Q}\underline{R}_2\right)\varepsilon\tau\sin\omega_1\tau - 2\omega_1\left(\underline{S}_1\underline{Q}\underline{R}_2\right) \\ \tilde{p}_{21} &= \left(\underline{S}_2\underline{H}\underline{Q}\underline{R}_1\right)\varepsilon\tau\cos\omega_2\tau \\ \tilde{k}_{21} &= \left(\underline{S}_2\underline{H}\underline{Q}\underline{R}_1\right)\varepsilon\tau\sin\omega_2\tau - 2\omega_2\left(\underline{S}_2\underline{Q}\underline{R}_1\right) \end{aligned} \tag{40}$$

Equations (38) and (39) represent a coupled system of two different variables $\pi_1(T_1)$ and $\pi_2^*(T_1)$ with matrix coefficient has complex entries. Its solution may be sought in the following form:

$$\pi_j\,(T_1) = \lambda_j e^{(\Xi + i\delta)T_1} \quad \text{and} \quad \pi_j^*\,(T_1) = \lambda_j^* e^{(\Xi - i\delta)T_1} \tag{41}$$

and the characteristic exponent Ξ at present resonance case is given by (28) except that Θ is replaced by the exponent Ξ and the coefficient complex matrix becomes:

$$\underline{P} = \begin{pmatrix} p_1 & \tilde{p}_{12} \\ \tilde{p}_{21} & p_2 \end{pmatrix} \qquad \underline{K} = \begin{pmatrix} k_1^+ + \delta & \tilde{k}_{12} \\ -\tilde{k}_{21} & k_2^- - \delta \end{pmatrix}$$

Stability criteria is the same as in the previous section in which the first condition for stability is $p_1+p_2>0$, while the critical condition that satisfied at this resonance case yields the following condition:

$$C_2\delta^2+2C_1\delta+C_0>0 \tag{42}$$

where the real coefficients C_2, C_1 and C_0 are:

$$\begin{aligned}
&C_2=2p_1p_2\\
&C_1=(p_2-p_1)\left(-\tilde{k}_{21}\tilde{p}_{12}+\tilde{p}_{21}\tilde{k}_{12}\right)+2p_1p_2\left(k_1^+-k_2^-\right)\\
&C_0=p_1p_2\left(p_1+p_2\right)^2+p_1p_2\left(k_1^+-k_2^-\right)^2-\left(-\tilde{k}_{21}\tilde{p}_{12}+\tilde{p}_{21}\tilde{k}_{12}\right)^2\\
&-\left(p_1-p_2\right)\left(k_1^+-k_2^-\right)\left(-\tilde{k}_{21}\tilde{p}_{12}+\tilde{p}_{21}\tilde{k}_{12}\right)-\left(p_1+p_2\right)^2\left(\tilde{k}_{21}\tilde{k}_{12}+\tilde{p}_{12}\tilde{p}_{21}\right)
\end{aligned} \tag{43}$$

The transition curves separating stable region from unstable one corresponding to

$$\begin{aligned}
\Omega&=\tfrac{1}{2}(\omega_1+\omega_2)+\varepsilon\left(-C_1+\sqrt{C_1^2-C_2C_0}\right)/C_2+...,\\
\Omega&=\tfrac{1}{2}(\omega_1+\omega_2)+\varepsilon\left(-C_1-\sqrt{C_1^2-C_2C_0}\right)/C_2+...
\end{aligned} \tag{44}$$

Clearly the resonance region lies between the two curves given by (44). The regions outside these curves represent the stable case. Similar results can be obtained for the case of Ω near $\frac{1}{2}(\omega_1-\omega_2)$ by changing the sign of ω_2 in the above analysis.

4. An extension for the 2-dimension Mathieu equation

Two dimension un-delayed Mathieu equations having complex damping term have been formulated in fluid mechanics for streaming flow in the 3-sratified weak viscous or weak viscoelastic fluids [13-15]. In what follows we extend the above mathematical procedure, in order to get the influence of the presence of the complex damping coefficients in the 2-dimension delay Mathieu equation. This extended is governed below, where equation (2) becomes:

$$\begin{aligned}
&\frac{d^2}{dt^2}\underline{X}(t)+\varepsilon\left(\underline{M}+i\underline{N}\right)\frac{d}{dt}\underline{X}(t)+\left[\underline{A}+i\varepsilon\underline{B}+4\varepsilon\underline{Q}\cos^2\Omega t\right]\underline{X}(t)\\
&=\varepsilon\underline{H}\underline{X}(t-\tau)
\end{aligned} \tag{45}$$

where $\underline{B}$, $\underline{M}$ and $\underline{N}$ are non-singular square matrices of 2×2 type. This system has a growth rate solution given by a quartic polynomial having complex coefficients. This characteristic equation leads to existing four complex roots (they are not necessary to be complex conjugates). The properties of this characteristic equation can be studied to achieve the stability behavior when the resonance influence is switch off. Therefore, Routh-Hurwitz criterion [25] for stability needs to apply in order to obtain the negativity of the real parts. On the other side, perturbation techniques are required in order to discuss the stability configuration near the resonance cases [3].

Consequently, the modified of equation (11) becomes:

$$\begin{aligned}
&\left(D_0^2\underline{I} + \underline{A}\right)\underline{X}_1(T_0, T_1) = -\left[\left(2i\omega_j + \varepsilon\tau\underline{H}e^{-i\omega_j\tau}\right)D_1 + i\omega_j\left(\underline{M} + i\underline{N}\right)\right.\\
&\left.-\underline{H}e^{-i\omega_j\tau} + i\underline{B} + 2Q\right]\underline{R}_j\pi_j(T_1)e^{i\omega_jT_0} - \left[\left(\varepsilon\tau\underline{H}e^{i\omega_j\tau} - 2i\omega_j\right)D_1\right.\\
&\left.-i\omega_j\left(\underline{M} + i\underline{N}\right) - \underline{H}e^{i\omega_j\tau} + i\underline{B} + 2\underline{Q}\right]\underline{R}_j\pi_j^*(T_1)e^{-i\omega_jT_0}\\
&-\underline{Q}\left(e^{i(\omega_j+2\Omega)T_0} + e^{i(\omega_j-2\Omega)T_0}\right)\underline{R}_j\pi_j(T_1)\\
&-\underline{Q}\left(e^{-i(\omega_j-2\Omega)T_0} + e^{-i(\omega_j+2\Omega)T_0}\right)\underline{R}_j\pi_j^*(T_1)
\end{aligned} \tag{46}$$

The removing of secular terms, in equation (46), at the non-resonant case leads to:

$$D_1\pi_j + \left(p_j^+ + ik_j^+\right)\pi_j = 0 \tag{47}$$

$$D_1\pi_j^* + \left(p_j^- + ik_j^-\right)\pi_j^* = 0 \tag{48}$$

They are having the following exponential solutions:

$$\pi_j(T_1) = \Lambda_j e^{-\left(p_j^+ + ik_j^+\right)T_1} \qquad \pi_j^*(T_1) = \Lambda_j^* e^{-\left(p_j^- + ik_j^-\right)T_1} \tag{49}$$

where the two parts given in the above exponentials are:

$$\begin{aligned}
p_j^\pm = {} & \underline{S}_j\left[\varepsilon\tau\underline{H}\left(2\underline{Q} \mp \omega_j\underline{N}\right)\cos\omega_j\tau \mp \varepsilon\tau\underline{H}\left(\underline{B} \pm \omega_j\underline{M}\right)\sin\omega_j\tau - \varepsilon\tau\underline{H}^2\right.\\
&\left.\pm 2\omega_j\left(\underline{B} \pm \omega_j\underline{M}\right) + 2\omega_j\underline{H}\sin\omega_j\tau\right]\underline{R}_j r
\end{aligned} \tag{50}$$

$$\begin{aligned}
k_j^\pm = {} & \underline{S}_j\left[\varepsilon\tau\underline{H}\left(\underline{B} \pm \omega_j\underline{M}\right)\cos\omega_j\tau + \left(\pm\varepsilon\tau\underline{H}\sin\omega_j\tau \mp 2\omega_j\underline{I}\right)\left(2\underline{Q} \mp \omega_j\underline{N}\right)\right.\\
&\left.\pm 2\omega_j\underline{H}\cos\omega_j\tau\right]\underline{R}_j
\end{aligned} \tag{51}$$

These solutions are in the form of the growth rate. Therefore, the Routh-Hurwitz criteria [25] for stability implies both the conditions:

$$p_j^+ > 0 \quad \text{and} \quad p_j^- > 0 \tag{52}$$

Note that, in the limiting case for un-damped two-dimension delayed Mathieu equation, the stability condition at the non-resonance case is the same as given by (25).

4.1. *The resonant case of Ω near ω_j where the damping coefficients are included*

When the complex damping coefficients are included the stability picture has been affected, while the mathematical procedure is the same as explained before. The stability criteria (33) is still satisfied and can be used to controls the stability behavior at the present case except that both the matrices $\underline{P}$ and $\underline{K}$ will modified to including the influence of the damping coefficients and they becomes:

$$\underline{P} = \begin{pmatrix} p_j^+ & \tilde{p}_j \\ \tilde{p}_j & p_j^- \end{pmatrix} \quad \text{and} \quad \underline{K} = \begin{pmatrix} \left(k_j^+ + \sigma\right) & \tilde{k}_j \\ -\tilde{k}_j & \left(k_j^- - \sigma\right) \end{pmatrix}$$

where $p_j^\pm$ and $k_j^\pm$ are as given by (50) and (51) while $\tilde{p}_j$ and $\tilde{k}_j$ are the same as in (29) and (30).

The first condition in (33) leads to $p_j^+ + p_j^- > 0$, which equivalent to the stability condition at the non-resonance case. Therefore, the critical condition for stability is the second one in (33). This condition has been formulated in terms of the detuning parameter σ as:

$$\begin{aligned} &2p_j^+p_j^-\sigma^2 + 4p_j^+p_j^-\left(k_j^+ - k_j^-\right)\sigma \\ &+\left[p_j^+p_j^-\left(p_j^+ + p_j^-\right)^2 + p_j^+p_j^-\left(k_j^+ - k_j^-\right)^2 - \left(p_j^+ + p_j^-\right)^2\left(\tilde{k}_j^2 + \tilde{p}_j^2\right)\right] > 0 \end{aligned} \tag{53}$$

This condition has two zeros, namely, σ_1 and σ_2.
In view of (23), the transition curves separating stable region from unstable one corresponding to:

$$\begin{aligned} \Omega &= \omega_j + \varepsilon\sigma_1 + ..., \\ \Omega &= \omega_j + \varepsilon\sigma_2 + ... \end{aligned} \tag{54}$$

The region lies between the tongs represent the unstable (resonance) case, which surrounding by the stable (the non-resonant) case.

4.2. *The resonant case of* Ω *near* $\frac{1}{2}(\omega_1 \pm \omega_2)$ *in the presence of the damping coefficients*

At the present case we obtain the following two coupled solvability conditions including the influence of the complex damping terms as well as the delay effects. The characteristic equation (43) is still satisfied except that the matrices $\underline{P}$ and $\underline{K}$ has been included the damping coefficients so that they forms as:

$$\underline{P} = \begin{pmatrix} p_1^+ & \tilde{p}_{12} \\ \tilde{p}_{21} & p_2^- \end{pmatrix} \qquad \underline{K} = \begin{pmatrix} k_1^+ + \delta & \tilde{k}_{12} \\ -\tilde{k}_{21} & k_2^- - \delta \end{pmatrix}$$

Stability criteria is the same as in the previous section in which the first condition for stability is $p_1^+ + p_2^- > 0$, while the critical condition that satisfied at present resonance case is the same form of condition (44). In the light of (37), we obtain the transition curves separating stable region from unstable one in the following form:

$$\begin{aligned} \Omega &= \tfrac{1}{2}(\omega_1 + \omega_2) + \varepsilon\left(-l_1 + \sqrt{l_1^2 - l_2 l_0}\right)/l_2 + ..., \\ \Omega &= \tfrac{1}{2}(\omega_1 + \omega_2) + \varepsilon\left(-l_1 - \sqrt{l_1^2 - l_2 l_0}\right)/l_2 + ... \end{aligned} \tag{55}$$

where the real coefficients l_2, l_1 and l_0 are:

$$\begin{aligned} &l_2 = 2p_1^+p_2^- \\ &l_1 = \left(p_2^- - p_1^+\right)\left(-\tilde{k}_{21}\tilde{p}_{12} + \tilde{p}_{21}\tilde{k}_{12}\right) + 2p_1^+p_2^-\left(k_1^+ - k_2^-\right) \\ &l_0 = p_1^+p_2^-\left(p_1^+ + p_2^-\right)^2 + p_1^+p_2^-\left(k_1^+ - k_2^-\right)^2 - \left(-\tilde{k}_{21}\tilde{p}_{12} + \tilde{p}_{21}\tilde{k}_{12}\right)^2 \\ &- \left(p_1^+ - p_2^-\right)\left(k_1^+ - k_2^-\right)\left(-\tilde{k}_{21}\tilde{p}_{12} + \tilde{p}_{21}\tilde{k}_{12}\right) - \left(p_1^+ + p_2^-\right)^2\left(\tilde{k}_{21}\tilde{k}_{12} + \tilde{p}_{12}\tilde{p}_{21}\right) \end{aligned}$$

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