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The Fastest, Simplified Method of Estimation of the Largest Lyapunov Exponent for Continuous Dynamical Systems with Time Delay

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This paper focuses on the applications of the new method of estimation of the Largest Lyapunov exponent. The method has been adapted to continuous dynamical systems with time delay. The paper presents efficiency of the new method in comparison with classical algorithms of LLE estimation. Computation times and convergence rates have been compared with the typically used method. It has been revealed in this paper that for the van der Pol oscillator, application of the new method increases the efficiency of calculations by 28% comparing to the classic one. Therefore, authors claim that the method presented in this paper is the fastest one in the assumed range of applications.

Keywords: largest Lyapunov exponent, continuous systems, time delay, estimation method, nonlinear dynamics.

1. Introduction

Depending on a dynamical system type and a kind of information that is useful for its investigations, different types of invariants characterizing system dynamics are applied. For instance one may use Kolmogorov entropy [1–2], correlation dimension [3–4], Lyapunov energy function [5] to determine stability of solutions and complexity of the system dynamics [6]. But when there occurs a need to predict the behavior of a real system with a possibility of different disturbances existence, Lyapunov exponents are one of the most commonly applied tools. That is because these exponents determine the exponential convergence or divergence of trajectories that start close to each other. The existence of such numbers has been proved by Oseledec theorem [7], but the first numerical study of the system's behavior using Lyapunov exponents had been done by Henon and Heiles [8], before the Oseledec theorem publication. The most important algorithms for calculating Lyapunov exponents for continuous systems have been developed by Benettin et al. [9] and Shimada and Nagashima [10], later improved by Benettin et al. [11] and Wolf [12]. One possible approach for systems with discontinuities or time delay, is estimation of Lyapunov exponents from a scalar time series basing on Takens procedure [13]. Numerical algorithms for such estimation have been developed by Wolf et al. [14], Sano and Sawada [15], and later improved by Eckmann et al. [16], Rosenstein et al. [17] and Parlitz [18]. Alternative method based on synchronization phenomena was elaborated by Stefanski [19–22].

Nowadays, Lyapunov Exponents are employed in many different areas of scientific research such as: materials [23–24], electric power systems [25], non-continuous systems [26-29, 20, 22], systems with time delay [21], aerodynamics [30], time series analysis [31–34], optimal control [35–36], chaotic encryption and secure communication [37–38], multi-objective optimization [39], parametric oscillations and fluctuating parameters [19, 40], neuronal models investigations [41].

Thus, there is still need to elaborate fast and simple methods of LE calculation. Recently, author of this paper presented a new, simple and effective method of estimation of the Largest Lyapunov Exponent (LLE) from the perturbation vector and its derivative dot product. It is based on simple computations involving only basic mathematical operations such as summing, subtracting, multiplying, dividing. It has been shown that LE can be extracted from information known before each integration step. The method can be applied in different aspects of the nonlinear systems control. Continuous systems [42], synchronization phenomena detection [43] and time series in control systems [44–46] have been investigated.

The article presents an adaptation of the previously described method to systems with time delay. The theoretical background has been explained. The method has been applied to estimate the LLE value of the forced van der Pol oscillator with a time delay. The bifurcation diagram and the corresponding LLE graph have been presented. The results are coincident with the values obtained in earlier research by means of a different LLE estimation method [21].

Moreover, the paper presents efficiency of the new method in comparison with classical algorithms of LLE estimation. The method has been tested for the forced van der Pol oscillator. Computation times and convergence rates have been compared with the method typically used [47] that involves calculations of perturbations lengths logarithms. The method presented in [47] uses the same approach as algorithms described in classical works, such as [9 - 11]. To authors' best knowledge, no faster algorithms of the LLE estimation for continuous dynamical systems than [9–11, 47] have been published. In this paper it has been revealed that for the van der Pol oscillator, application of our new method increases the efficiency of calculations by 28% comparing to [47]. Therefore, authors claim that the method presented in this paper is the fastest one in the assumed range of applications.

2. The method

Assume that a dynamical system is described by the set of differential equations in the form:

$$\frac{dx}{dt} = f(x) \tag{1}$$

where x – state vector and f is a vector field that (in general) depends on x. Evolution of a small perturbation z near any point x in such system can be found from the equation:

$$\frac{dz}{dt} = \frac{df}{dx}(x)z \tag{2}$$

where $\frac{df}{dx}(x)$ is the Jacobi matrix obtained by differentiation of $f$with respect to x. As it has been shown in the article [42], the value of LLE can be estimated from the following expression:

$$\lambda^* = \frac{\frac{dz}{dt}z}{|z|^2} \tag{3}$$

where z is a perturbation vector, whose evolution can be obtained by numerical integration of equation (2). The approximate value of the LLE (λ) is obtained by averaging values of λ^* from subsequent computation steps. For long enough time of integration, the average value of λ^* converges to the LLE.

However, it must be noted that formulas (1-2) are no longer valid when a time delay is present in the system. A dynamical system with a constant time delay can be, in general, described by the set of differential equations in the form:

$$\frac{dx}{dt} = g(x(t), x(t-\tau)) \tag{4}$$

where τ is a constant time delay. Consider two trajectories that start infinitesimally close to each other: an undisturbed one $x(t)$ and a disturbed one $y(t) = x(t) + z(t)$ where $z(t)$ is an infinitesimal perturbation. In such case, evolution of the perturbation $z(t)$ is defined by the following differential equation:

$$\begin{aligned} &\tfrac{dz}{dt} = g(y(t), y(t-\tau)) - g(x(t), x(t-\tau)) \\ &\approx \tfrac{\partial g}{\partial x(t)}(x(t))z(t) + \tfrac{\partial g}{\partial x(t-\tau)}(x(t-\tau))z(t-\tau) \end{aligned} \tag{5}$$

From formula (5) one can notice that evolution of a perturbation in the system (4) depends on current state $x(t)$, current value of perturbation $z(t)$, delayed state $x(t-\tau)$ and delayed perturbation $z(t-\tau)$. It has been checked by numerical simulations that when evolution of the perturbation is computed according to (5), formula (3) can be applied to estimate the value of the LLE of the system (4). Please note that the numerical procedure must involve memory of the previous states and previous perturbations. It has to be taken into account that normalization of the perturbation $z(t)$ must be conducted along with division of all the previous perturbations in the memory by the same factor.

3. Numerical simulations

In the first part of the numerical experiment, the delayed van der Pol oscillator has been analyzed. The differential equation of such system is as follows:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \beta x^3 = \kappa x(t-\tau) - q\sin(\omega t) \tag{6}$$

where $\mu, \beta, \kappa, \tau, \omega$ are constant parameters. In the state-space form (1), the system (6) can be presented as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \mu(1-x_1^2)x_2 - \beta x_1^3 + \kappa x_1(t-\tau) - q\sin(x_3) \\ \dot{x}_3 &= \omega \end{aligned} \tag{7}$$

Please note that delayed function has been described explicitly with the argument $(t-\tau)$, whereas all other functions are implicitly referred to the current time (t). Evolution of a perturbation in the system (6) is described by the following equation:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= (-2\mu x_1 x_2 - 3\beta x_1^2)z_1 + \mu(1-x_1^2)z_2 + \kappa z_1(t-\tau) \\ \dot{z}_3 &= 0 \end{aligned} \tag{8}$$

It has been assumed that the third state variable (forcing phase) is not disturbed. Thevalue κ has been used as a control parameter of the bifurcation diagram.

The simulation script and the LLE estimation procedure have been created using Python 3 with NumPy and SciPy packages. Integration has been performed by means of Runge–Kutta method implemented in the SciPy package. The maximum integration step equal to $\Delta t = 10^{-3}$ has been selected.

In order to simulate the system (7), states $x(t-\Delta t), x(t-2\Delta t), ..., x(t-\tau)$ and perturbations $z(t-\Delta t), z(t-2\Delta t), ..., z(t-\tau)$ must be temporarily saved in the memory. In the simulation under consideration, previous states and perturbations have been stored in an array of vectors. To find the value of a state or a perturbation in the mid step, the linear approximation has been used.

Situations in which the length of a perturbation attains too high or too low values must be avoided. Therefore, perturbation length control has been implemented. When the length of a perturbation gets out from a desired range, it is normalized to 1. Moreover, all the previous perturbations from the program memory are divided by the same factor. In the presented simulations, normalization is triggered when the magnitude of a perturbation was above 10^6 or below 10^{-6}.

After every integration step value λ^* is calculated according to the formula (3). Throughout the simulation, the averaged value λ approaches the true value of the LLE.

The second part of the experiment has been focused on efficiency comparison between the presented method of the LLE estimation and the classical method [47]. The test involved estimation of the LLE value of the forced van der Pol oscillator described by the following equation:

$$\ddot{x} - \mu(1-x^2)\dot{x} + x = q\sin(\omega t) \tag{9}$$

In the state space form (1), the system (9) can be presented as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \mu(1-x_1^2)x_2 - x_1 + q\sin(x_3) \\ \dot{x}_3 &= \omega \end{aligned} \tag{10}$$

Evolution of a perturbation in the system (10) is described by the following equation:

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= (-2\mu x_1 x_2 - 1)z_1 + \mu(1 - x_1^2)z_2 \\
\dot{z}_3 &= 0
\end{aligned} \tag{11}$$

Again, it has been assumed that the third state variable (forcing phase) is not disturbed. The value μ has been used as a control parameter of the bifurcation diagram.

Simulation programs have been developed in C++ using CodeBlocks 15.12. In order to integrate the system of differential equations (10-11), Runge-Kutta method of the fourth order has been used. The integration step equal to $\Delta t = 10^{-3}$ has been selected.

In order to avoid situations in which the length of a perturbation attains too high or too low values, perturbation length control was implemented. When the length of a perturbation gets out from a desired range, it is normalized to 1. In the presented simulations, normalization is triggered when the magnitude of a perturbation was above 10^{10} or below 10^{-10}. After every integration step, value λ^* is calculated according to the formula (3). Throughout the simulation, the averaged value λ approaches the true value of the LLE.

The calculations must be stopped when the estimated value of the LLE stabilizes. Therefore, the values of λ computed in subsequent iterations of the procedure are stored in a buffer. When the buffer is full, the standard deviation of all the values in the buffer is calculated. If the standard deviation is low enough, the estimated LLE is considered stable. In such case, the average of all the values in the buffer is returned as the final LLE. On the other hand, when the standard deviation in the buffer is higher than the required threshold, the buffer is cleared and the calculations are continued.

For the purpose of time comparison, a modification of the simulation program has been created. It used the standard method of LE estimation based on calculation of natural logarithm of the perturbation length [47]. Most of the program remained unchanged in order to reduce the undesired influences on the method efficiency measurement. Both programs were run separately on the same computer. Authors did their best to provide equal conditions in which two versions of the software were executed. In particular, the computer was disconnected from the Internet while calculating. All the unnecessary processes were switched off, simulation software was executed with the highest possible priority ("realtime").

4. Results of numerical simulations

Firstly, the LLE values have been estimated for the system (7) in the range of control parameter κ from 0.0 to 9.0 with the step of 0.01. The values of other parameters have been assumed as follows: $\mu = 0.2$, $\beta = 1.0$, $\tau = 2.0, q = 17.0$, $\omega = 4.0$. The same numbers have been used in the reference [21]. The system has been run with zero initial conditions. Components z_1 and z_2 of the initial perturbation $z(0)$ have been selected randomly in the range [0,1). Due to the assumption of the undisturbed forcing phase, the third component of the initial perturbation has been set to 0. It has been assumed that $z = 0$ and $x = 0$ for $t < 0$.

The bifurcation diagram of the state variable x_1 of the delayed van der Pol's equations (7) together with the corresponding graph of the LLE calculated using the new method are depicted in the Fig. 1.

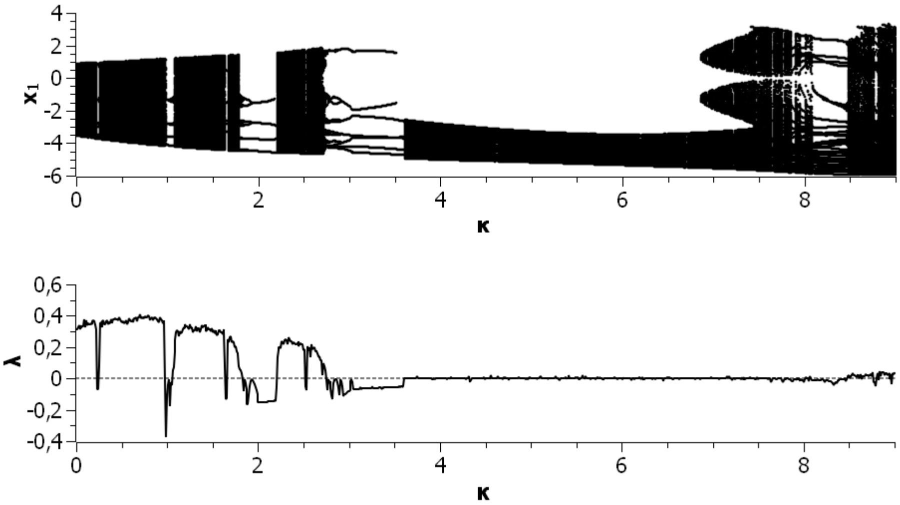

Figure 1 Bifurcation diagram and the LLE graph of the delayed van der Pol system (7)

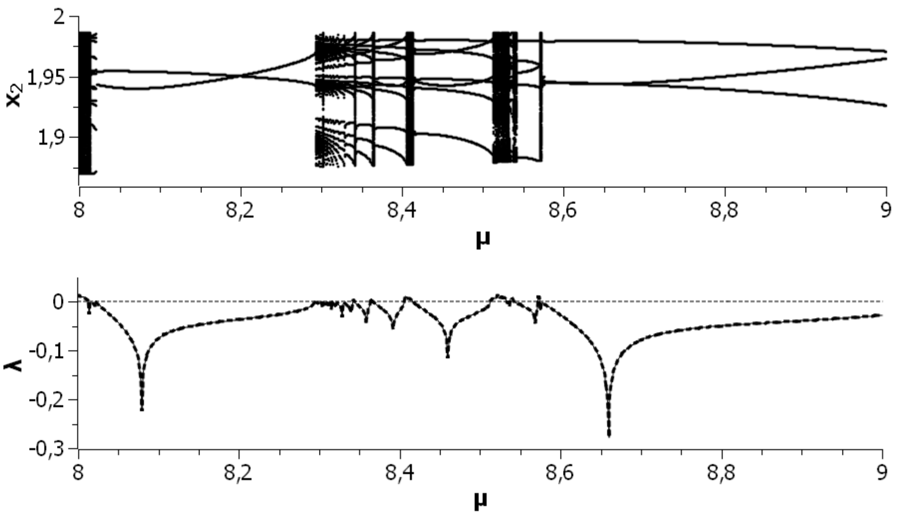

Figure 2 Bifurcation diagram and the LLE graph of the forced van der Pol system (10)

The Largest Lyapunov exponent for the forced van der Pol oscillator (10) has been computed with the following parameters values: $q = 1.2$, $\omega = \frac{2\pi}{10}$. The parameter μ has been used as the control parameter in the range from 8.0 to 9.0 with the step of 0.001. In the simulation, a unit random vector has been used as the initial conditions. The initial perturbation has also been selected randomly and normalized to 1.

The bifurcation diagram of the state variable x_2 of the van der Pol oscillator (10) together with the corresponding graph of the LLE is depicted in the Fig. 2. The LLE graph obtained by means of the new method has been drawn with the solid line, whereas values of the LLE calculated using the classical method are marked with the dotted line. It can be noticed that both methods yield almost the same results.

5. Computation time comparisons

Comparison of the computation time has been performed for the forced van der Pol system (10) only. For each value of the control parameter μ, the time of the LLE estimation has been measured from the beginning of calculations till the stabilization of the LLE. Further on, for each μ, the ratio of the calculations time with the new method to the calculations time with the classical method has been computed. The average value of the ratio is approximately equal to 0.718 with the standard deviation 0.005. This means that for the van der Pol oscillator, the new method is, in the average, 28% faster than the classical one. To authors' best knowledge, there is no faster algorithm of the LLE estimation for continuous dynamical systems than the one used for comparison. Therefore,authors claim that the method presented in this paper is the fastest one in the assumed range of applications.

6. Conclusions

The first part of the experiment confirmed that the presented method of the LLE estimation can be applied to systems with time delay. The results obtained by means of the new method are coincident with values obtained from the synchronization method [21]. Therefore, using the presented method, calculation of the LLE of the systems with time delay can be significantly simplified.

The second part of the experiment showed that the presented method of the LLE estimation is, in the average, 28% faster than the classical one for the van der Pol oscillator. Therefore, to authors' best knowledge, the presented method is the fastest algorithm of the LLE estimation for continuous dynamical systems.

In the future it is planned to apply the method to calculations of the whole Lyapunov Exponents spectrum and to adopt it to systems with discontinuities.

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References

[1] **Bennettin, G., Froeschle, C., Scheidecker, J. P.**: Kolmogorov entropy of a dynamical system with increasing number of degrees of freedom, *Phys. Rev.*, A, 19, 2454–2460, **1979**.

[2] **Benettin, G., Galgani, L., Strelcyn, J. M.**: Kolmogorov entropy and numerical experiment, *Phys. Rev.*, A, 14, 2338– 2345, **1976**.

[3] **Grassberger, P., Procaccia, I.**: Characterization of strange attractors, *Phys. Rev. Lett.*, 50, 346–349, **1983**.

[4] **Alligood, K. T., Sauer, T. D., Yorke, J. A.**: Chaos an Introduction to Dynamical Systems, *Springer, New York*, **2000**.

[5] **Zhang, Y., Chen, D., Guo, D., Liao, B., Wang, Y.**: On exponential convergence of nonlinear gradient dynamics system with application to square root finding, *Nonlinear Dynamics*, 79, 983–1003, **2015**.

[6] **Eckmann, J-P., Ruelle, D.**: Ergodic theory of chaos and strange attractors, *Rev. Mod. Phys.*, 57, 617–656, **1985**.

[7] **Oseledec, V. I.**: A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems, *Trans. Mosc. Math. Soc.*, 19, 197–231, **1968**.

[8] **Henon, M., Heiles, C.**: The applicability of the third integral of the motion: some numerical results, *Astron. J.*, 69, 77, **1964**.

[9] **Benettin, G., Galgani, L., Giorgilli, A., Strelcyn, J. M.**: Lyapunov exponents for smooth dynamical systems and Hamiltonian systems; a method for computing all of them, part I: theory, *Meccanica*, 15, 9–20, **1980**.

[10] **Shimada, I., Nagashima, T.**: A numerical approach to ergodic problem of dissipative dynamical systems, *Prog. Theor. Phys.*, 61, 6, 1605–1616, **1979**.

[11] **Benettin, G., Galgani, L., Giorgilli, A., Strelcyn, J. M.**: Lyapunov exponents for smooth dynamical systems and Hamiltonian systems; a method for computing all of them, part II: numerical application, *Meccanica*, 15, 21–30, **1980**.

[12] **Wolf, A.**: Quantifying chaos with Lyapunov exponents. In: Holden, V. (ed.), *Chaos. Manchester University Press, Manchester*, 273–290, **1986**.

[13] **Takens, F.**: Detecting strange attractors in turbulence, *Lect. Notes Math.*, 898, 366, **1981**.

[14] **Wolf, A., Swift, J. B., Swinney, H. L., Vastano, J. A.**: Determining Lyapunov exponents from a time series, *Physica*, D, 16, 285–317, **1985**.

[15] **Sano, M., Sawada, Y.**: Measurement of the Lyapunov spectrum from a chaotic time series, *Phys. Rev. Lett.*, 55, 1082– 1085, **1985**.

[16] **Eckmann, J. P., Kamphorst, S. O., Ruelle, D., Ciliberto, S.**: Lyapunov exponents from a time series, *Phys. Rev. Lett.*, 34, 9, 4971–4979, **1986**.

[17] **Rosenstein, M. T., Collins, J. J., De Luca, C. J.**: A practical method for calculating largest Lyapunov exponents from small data sets, *Physica*, D, 65, 1–2, 117–134, **1993**.

[18] **Parlitz, U.**: Identification of true and spurious Lyapunov exponents from time series, *Int. J. Bifurc. Chaos*, 2, 1, 155–165, **1992**.

[19] **Stefanski, A.**: Lyapunov exponents of the systems with noise and fluctuating parameters, *J. Theor. Appl. Mech.*, 46, 3, 665–678, **2008**.

[20] **Stefański, A.**: Estimation of the largest Lyapunov exponent in systems with impacts, *Chaos Solitons Fractals*, 11, 15, 2443–2451, **2000**.

[21] **Stefański, A., Dąbrowski, A., Kapitaniak, T.**: Evaluation of the largest Lyapunov exponent in dynamical systems with time delay, *Chaos Solitons Fractals*, 23, 1651–1659, **2005**.

[22] **Stefański, A., Kapitaniak, T.**: Estimation of the dominant Lyapunov exponent of non-smooth systems on the basis of maps synchronization, *Chaos Solitons Fractals*, 15, 233–244, **2003**.

[23] **Iwaniec, J., Uhl, T., Staszewski, W.**: Detection of changes in cracked aluminium plate determinism by recurrence analysis, *Nonlinear Dynamics*, 70, 1, 125–140, **2012**.

[24] **Rybaczuk, M., Aniszewska, D.**: Lyapunov type stability and Lyapunov exponent for exemplary multiplicative dynamical systems, *Nonlinear Dynamics*, 54, 345–354, **2008**.

[25] **Wadduwage, D. P., Qiong Wu, C., Annakkage, U. D.**: Power system transient stability analysis via the concept of Lyapunov Exponents, *Electric Power Systems Research*, 104, 183–192, **2013**.

[26] **Serweta, W., Okolewski, A., Błażejczyk-Okolewska, B., Czolczynski, K., Kapitaniak, T.**: Mirror hysteresis and Lyapunov exponents of impact oscillator with symmetrical soft stops, *International Journal of Mechanical Sciences*, 101–102, 89–98, **2015**.

[27] **Serweta, W., Okolewski, A., Błażejczyk-Okolewska, B.**: Lyapunov exponents of impact oscillators with Hertz's and Newton's contact models, *International Journal of Mechanical Sciences*, 89 , 194–206, **2014**.

[28] **Lamarque, C. H., Malasoma, J. M.**: Analysis of nonlinear oscillations by wavelet transform: Lyapunov exponents, *Nonlinear Dynamics*, 9, 333–347, **1996**.

[29] **Yue, Y., Xie, J., Gao, X.**: Determining Lyapunov spectrum and Lyapunov dimension based on the Poincare map in a vibro-impact system, *Nonlinear Dynamics*, 69, 743–753, **2012**.

[30] **Hu, D. L., Huang, Y., Liu, X. B.**: Moment Lyapunov exponent and stochastic stability of binary airfoil driven by non-Gaussian colored noise, *Nonlinear Dynamics*, 70, 1847–1859, **2012**.

[31] **Yang, C., Wu, C. Q., Zhang, P.**: Estimation of Lyapunov exponents from a time series for n-dimensional state space using nonlinear mapping, *Nonlinear Dynamics*, 69, 1493–1507, **2012**.

[32] **Yang, C., Wu, C. Q.**: On stability analysis via Lyapunov exponents calculated from a time series using nonlinear mapping—a case study, *Nonlinear Dynamics*, 59, 239–257, **2010**.

[33] **Sun, Y., Wu, C. Q.**: A radial-basis-function network-based method of estimating Lyapunov exponents from a scalar time series for analyzing nonlinear systems stability, *Nonlinear Dynamics*, 70, 1689–1708, **2012**.

[34] **Yang, C., Wu, C. Q.**: A robust method on estimation of Lyapunov exponents from a noisy time series, *Nonlinear Dynamics*, 64, 279–292, **2011**.

[35] **Zhu, W. Q.**: Feedback Stabilization of Quasi Nonintegrable Hamiltonian Systems by Using Lyapunov Exponent, *Nonlinear Dynamics*, 36, 2, 455–470, **2004**.

[36] **Zhu, W. Q., Huang, Z. L.**: Stochastic Stabilization of Quasi-Partially Integrable Hamiltonian Systems by Using Lyapunov Exponent, *Nonlinear Dynamics*, 33, 2, 209–224, **2003**.

[37] **Li, C., Wang, J., Hu, W.**: Absolute term introduced to rebuild the chaotic attractor with constant Lyapunov exponent spectrum, *Nonlinear Dynamics*, 68, 4, 575–587, **2012**.

[38] **Li, S. Y., Huang, S. C., Yang, C. H.**: Generating tri-chaos attractors with three positive Lyapunov exponents in new four order system via linear coupling, *Nonlinear Dynamics*, 69, 3, 805–816, **2012**.

[39] **Fraga, L. G., Tlelo-Cuautle, E.**: Optimizing the maximum Lyapunov exponent and phase space portraits in multi-scroll chaotic oscillators, *Nonlinear Dynamics*, 76, 1503–1515, **2014**.

[40]] **Rong, H., Meng, G., Wang, X**.: Invariant Measures and Lyapunov Exponents for Stochastic Mathieu System, *Nonlinear Dynamics*, 30, 4, 313–321, **2002**.

[41] **Soriano, D. C., Fazanaro, F. I., Suyama, R.**: A method for Lyapunov spectrum estimation using cloned dynamics and its application to the discontinuously-excited FitzHugh–Nagumo model, *Nonlinear Dynamics*, 67, 1, 413–424, **2012**.

[42] **Dąbrowski, A.**: Estimation of the largest Lyapunov exponent from the perturbation vector and its derivative dot product, *Nonlinear Dynamics*, 67, 1, 283–291, **2012**.

[43] **Dąbrowski, A.**: The largest transversal Lyapunov exponent and master stability function from the perturbation vector and its derivative dot product (TLEVDP), *Nonlinear Dynamics*, 69, 3, 1225–1235, **2012**.

[44] **Balcerzak, M., Dąbrowski, A., Kapitaniak, T., Jach, A.**: Optimization of the Control System Parameters with Use of the New Simple Method of the Largest Lyapunov Exponent Estimation, *Mechanics and Mechanical Engineering*, 17, 3, 225–239, **2013**.

[45] **Pijanowski, K., Dąbrowski, A., Balcerzak, M.**: New method of multidimensional control simplification and control system optimization, *Mechanics and Mechanical Engineering*, 19 2, 127–139, **2015**.

[46] **Dąbrowski, A.**: Estimation of the the Largest Lyapunov exponent-like (LLEL) stability measure parameter from the perturbation vector and its derivative dot product (part 2) experiment simulation, *Nonlinear Dynamics*, 78, 3, 1601–1608, **2014**.

[47] **Parker, T. S., Chua, L. O.**: Practical Numerical Algorithms for Chaotic Systems, *Springer-Verlag*, Berlin, **1989**.