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Swing-Up Problem of an Inverted Pendulum – Energy Space Approach

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This paper describes a novel, energy space based approach to the swing-up of an inverted pendulum. The details of the swing-up problem have been described. Equations of the velocity-controlled have been presented. Design of the controller based on energy space notion has been elaborated. The control algorithm takes into account state constraints and control signal constraints. Parameters of the controller have been optimized by means of the Differential Evolution method. A numerical simulation of the inverted pendulum driven by the proposed controller has been conducted, its results have been presented and elaborated. The paper confirms that the proposed method results in a simple and effective swing-up algorithm for a velocity-controlled inverted pendulum with state constraints and control signal constraints.

Keywords: Swing-up, inverted pendulum, controller, state constraints, control constraints, energy space.

1. Introduction

The inverted pendulum is a kind of pendulum in which the axis of rotation is fixed to a cart. The cart is able to move along the horizontal axis in a controlled way. Among classical examples of the control theory applications, the inverted pendulum plays a unique role. Such device can be treated as a model of a space booster on takeoff – the problem of space booster attitude control is similar to the problem of the inverted pendulum stabilization [1]. Such device can be treated as a benchmark of various control strategies.

The swing-up problem of the inverted pendulum is to find a control of the cart drive that swings the pendulum from the downright position to the upright vertical position. Different approaches to solve this problem have been proposed. One can use the energy-based controller [2]. Such solution offers simple and efficient swing-up algorithm for force-controlled inverted pendulum without state or control constraints. Another approach is to apply the optimal control theory [3]. It enables to find an open-loop control that drives a dynamical system under state and control constraints from one state to another in an optimal way (with respect to a selected optimality criterion). However, design of a feedback controller in such a manner can be problematic.

In this paper the energy approach, based on the notion of energy space [4, 5], has been used to design a swing-up driver of a velocity-controlled inverted pendulum. The state and control constraints have been taken into account. The control signal depends on the feedback, so the designed control system has a closed-loop structure. The efficiency of the proposed algorithm has been proven by numerical simulations of the inverted pendulum model with realistic parameters. Coefficients of the controller have been found by means of the Differential Evolution optimization algorithm [6]. It has been shown that this novel approach results in design of a simple and effective swing-up controller for a real, velocity-controlled inverted pendulum with state and control constraints.

2. The energy space swing-up algorithm

This paper is focused on the swing-up problem of an inverted pendulum (Fig. 1). The inverted pendulum is a kind of pendulum in which the axis of rotation is fixed to a cart. The cart is able to move along the horizontal axis x in a controlled way. The swing-up problem is to find a control of the cart drive that changes the bar angle $\alpha(t)$ from $\alpha(0) = \pi$ to $\alpha(t_f) = 0$, where t_f is an unknown final time.

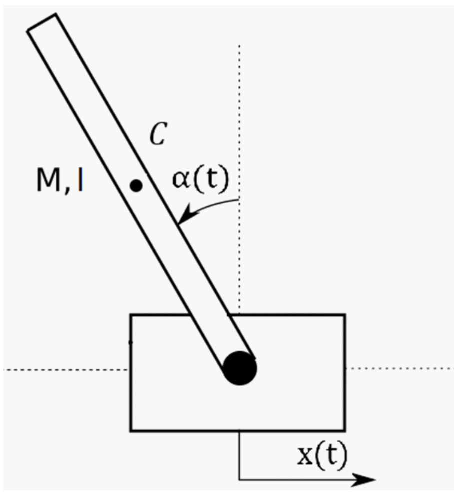

Figure 1 Sketch of the considered control object – the inverted pendulum

It has been assumed that the pendulum's drive is velocity-controlled. It means that the control signal $u(t)$ supplied to the drive is equal to the desired velocity of the cart. If the drive is stiff enough, then the motion of the pendulum's bar does not influence position of the cart *x(t)*. Providing that the drive can be approximated by a linear differential equation of the first order, the dependence between acceleration of the cart and the control signal is as follows:

$$\ddot{x}(t) = a[u(t) - \dot{x}(t)] \tag{1}$$

where *u(t)* is the control signal and a is a drive constant, which can be determined in the identification process.

The equation of motion of the inverted pendulum can be easily derived using Lagrange approach [7]. Assume that the pendulum's bar is uniform, its mass center C is in the middle of its length and it is loaded by a friction torque $\tau * ml^2/3$. Then, the following equation of motion (2) is obtained:

$$\ddot{\alpha}(t) = \frac{3g}{2l}\sin(\alpha(t)) + \frac{3\ddot{x}(t)}{2l}\cos(\alpha(t)) - \tau(\dot{\alpha}(t)) \tag{2}$$

where l is the length of the bar. Equations (2) and (3) constitute a complete mathematical description of the inverted pendulum.

The total mechanical energy of the bar is as follows (3):

$$E = \frac{mgl}{2}[\cos(\alpha(t)) - 1] + \frac{ml^2}{6}[\dot{\alpha}(t)]^2 \tag{3}$$

It can be noticed that *E =0* if and only if the pendulum bar remains in an upright vertical position (both $\alpha(t)$ and its derivative equal to 0). Therefore, the main goal of the swing-up controller is to drive the total mechanical energy E to the value 0.

To analyze the energy flow in the system, the notion of energy space can be applied [4, 5]. In this approach, the total mechanical energy is represented by a vector. Each coordinate of such vector is connected with the potential or kinetic energy of one body in the system. The vector is constructed in such a way that its squared length is equal to the total mechanical energy in the system.

The energy space vector for the system under consideration can be constructed in the following manner. Let $\mathbf{v}$ be the state vector of the pendulum's bar (4):

$$\mathbf{v}(t) = [v_1, v_2]^T = [\alpha(t), \dot{\alpha}(t)]^T \tag{4}$$

In the state space, equations of motion of the bar are as follows (5):

$$\dot{\mathbf{v}}(t) = [v_2, \frac{3g}{2l}\sin(v_1) + \frac{3\ddot{x}(t)}{2l}\cos(v_1) - \tau(v_2)]^T \tag{5}$$

Note that the expression $[cos(\alpha(t)) - 1]$ from the Eq. (3) can be transformed using the Taylor series expansion [8]. By substituting $v_1$in the place of $\alpha(t)$ one obtains (6):

$$\cos(\alpha(t)) - 1 = -\frac{v_1^2}{2!} + \frac{v_1^4}{4!} - ... = \frac{v_1^2}{2}(-1 + \frac{2v_1^2}{4!} - ...) = \frac{v_1^2}{2}b^2(v_1) \tag{6}$$

where $b^2(v_1)$ is defined by the expression in the brackets. Obviously, $b(v_1)$ may be a complex number. Using the Eq. (3, 4, 6), the expression for the total mechanical energy (3) can be presented in the following form (7):

$$E = \frac{mgl}{2}b^2(v_1)\frac{v_1^2}{2} + \frac{ml^2}{3}\frac{v_2^2}{2} \tag{7}$$

Therefore, the energy vector $\mathbf{v}_e$[4] can be defined as follows (8):

$$\mathbf{v}_e(t) = \begin{bmatrix} v_{e1} \\ v_{e2} \end{bmatrix} = \begin{bmatrix} b(v_1)\sqrt{\frac{mgl}{2}} & 0 \\ 0 & \sqrt{\frac{ml^2}{3}} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{A}(v_1)\mathbf{v}(t) \tag{8}$$

where $\mathbf{v}_e$ is the energy vector and the matrix $\mathbf{A}(v_1)$ is used to transform a vector from the state space to the energy space. According to [4], the energy product of two vectors is defined as in the Eq. (9):

$$< \mathbf{u}, \mathbf{w} >= \frac{1}{2}(u_1 w_1 + u_2 w_2 + ... + u_n w_n) \tag{9}$$

where n is the dimension of the vectors $\mathbf{u}$, $\mathbf{w}$. The norm of the energy vector $\mathbf{v}_e$ is obtained as in the Eq. (10):

$$|\mathbf{v}_e| = \sqrt{< \mathbf{v}_e, \mathbf{v}_e >} \tag{10}$$

Using the definitions (9,10) it can be easily confirmed that $|\mathbf{v}_e|^2 = E$.

To observe the evolution of $\mathbf{v}_e$its time derivative must be known. Differentiation of the Eq. (8) yields (11):

$$\dot{\mathbf{v}}_e(t) = \begin{bmatrix} [b(v_1) + v_1 b'(v_1)]\sqrt{\frac{mgl}{2}} & 0 \\ 0 & \sqrt{\frac{ml^2}{3}} \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \mathbf{B}(v_1)\dot{\mathbf{v}}(t) \tag{11}$$

The objective of the swing-up controller is to drive the total mechanical energy of the pendulum bar to the value 0. Therefore, in the case when $E < 0$, the control should increase the total energy. On the other hand, when $E > 0$, the control is supposed to decrease the total energy. To find the proper control, the time derivative of E (12) must be analyzed:

$$\dot{E} = \frac{d}{dt}|\mathbf{v}_e|^2 =< \mathbf{v}_e, \dot{\mathbf{v}}_e >=< \mathbf{v}A(v_1)\mathbf{v}(t), \mathbf{v}B(v_1)\dot{\mathbf{v}}(t) > \tag{12}$$

The value of the time derivative of E can be derived by substituting formulas (5,8,11) into (12). From the control point of view, the most important part of the result from (12) is the component containing the cart's acceleration term, which directly depends on the control $u(t)$ according to the drive model (1). Taking this fact into account, the derivative of E can be presented in the following form (13):

$$\dot{E} = \frac{ml}{2} v_2 \cos(v_1)\ddot{x}(t) + R(\mathbf{v}(t)) = \frac{ml}{2} v_2 \cos(v_1) a[u(t) - \dot{x}(t)] + R(\mathbf{v}(t)) \tag{13}$$

where $R(\mathbf{v}(t))$ is the part of the result which does not depend on the acceleration of the cart and, consequently, on the control.

Taking into account that the drive constant a is a positive real number, the following conclusions can be drawn from (13). If the control $u(t)$ has the same sign as the product $v_2 \cos(v_1)$, then the value of energy derivative is increased. On the other hand, if signs of $u(t)$ and $v_2 \cos(v_1)$ are different, then the value of energy derivative is decreased. Therefore, in order to increase energy of the system, signs of $u(t)$ and $v_2 \cos(v_1)$ should be kept the same. To decrease energy of the system, opposite signs of $u(t)$ and $v_2 \cos(v_1)$ have to be maintained.

In any real drive, the attainable velocities are limited. Therefore, the signal $u(t)$ must be bounded. Assume that for any time t the control signal $u(t)$ remains in the range of permissible values $[-u_{\max}, u_{\max}]$, where the constant $u_{\max} > 0$ is the maximal absolute value of the control signal. In such situation, the fastest increase (or decrease) of the total energy is assured when the control *u(t)* attains only the extreme values: $-u_{\max}$ or $u_{\max}$. Such control strategy is called "bang-bang control" [3]. It can be proven that the minimum-time swing-up control of the inverted pendulum involves application of the "bang-bang" strategy. This fact can results from the Pontryagin Minimum Principle [3].

Although the bang-bang control can lead to the fastest transition of the system from one state to another, it may also cause a few practical problems. First of all, such strategy can lead to unwanted high-frequency switching of the control signal $u(t)$ between the extreme values: $-u_{\max}$, $u_{\max}$. Secondly, in case of the inverted pendulum, this control may not lead to proper stabilization of the bar near the upright vertical position $\alpha(t) = 0$.

Moreover, state constraints have to be taken into account. For example, motion of the cart in a typical inverted pendulum is physically constrained. Therefore, position of the cart $x(t)$ has to remain in the interval $[-x_{\max}, x_{\max}]$ for a positive constant $x_{\max}$. If the position of the cart reaches an extreme, then the control should either invert direction of motion of the cart, or stop it.

Summing up, the following control strategy is proposed:

$$\hat{u}(t) = \begin{cases} u_{\max} \quad if \quad t < t_s \\ u_{\max} sgn[v_2 \cos(v_1)] \quad if \quad |v_2 \cos(v_1)| \geq \delta, |v_1| \geq \alpha_0, t \geq t_s \\ 0 \quad if \quad |v_2 \cos(v_1)| < \delta, |v_1| \geq \alpha_0, t \geq t_s \\ -(k_1 v_1 + k_2 v_2 + k_3 x + k_4 \dot{x}) \quad if \quad |v_1| < \alpha_0, t \geq t_s \end{cases} \tag{14}$$

$$u(t) = \begin{cases} \max\{-u_{\max}, \min[u_{\max}, \hat{u}(t)]\} \quad if \quad |x| < x_{\max} \\ \max\{0, \min[u_{\max}, \hat{u}(t)]\} \quad if \quad x < -x_{\max} \\ \max\{-u_{\max}, \min[u_{\max}, \hat{u}(t)]\} \quad if \quad x > x_{\max} \end{cases} \tag{15}$$

This control strategy works as follows. Assume that the inverted pendulum is started when the bar remains in downright vertical position $\alpha(0) = \pi$ and the initial position of the cart is $x(0) = 0$. At the beginning, until the starting time t_s, the control signal attains constant, maximal value in order to provide an initial energy to the system. After the starting time t_s, the control tends to increase the total energy E by selecting extreme values $-u_{\max}$, $u_{\max}$ according to sign of the product $v_2 \cos(v_1)$. However, in order to avoid high-frequency switching when the sign of $v_2 \cos(v_1)$ changes, a dead-zone has been introduced: the control signal is set to an extreme value only if the absolute value of the product $v_2 \cos(v_1)$ attains a minimum equal to δ. Otherwise, the control is set to 0. When the pendulum bar is close enough to the upright vertical position ($|\alpha(t)| < \alpha_0$, the linear control is switched on in order to assure proper stabilization of the bar. The constant α_0 is the pendulum angle at which the linear control is switched on. Values of parameters t_s, δ, α_0 must be determined experimentally. It is assumed that once the linear control is switched on, it remains on even if the absolute value of the pendulum angle $\alpha(t)$ happens to exceed α_0. Finally, the state constraint is imposed on the control. If the cart's position $x(t)$ is beyond the interval $[-x_{\max}, x_{\max}]$, then, according to (15), value of the control signal is limited in order to prevent the cart from moving away any further.

3. The numerical simulation

The control strategy defined by the formulas (14-15) has been applied for the inverted pendulum described by the equations (1-2). It has been assumed that friction at the bearings is described by the sum of linear and quadratic terms that represent viscous friction and air drag force respectively (16):

$$\tau(\dot{\alpha}) = c_1\dot{\alpha} + c_2(\dot{\alpha})^2 sgn(\dot{\alpha}) \tag{16}$$

Assume the complete state-space vector of the inverted pendulum in the form (17):

$$\mathbf{V} = [v_1, v_2, v_3, v_4] = [\alpha, \dot{\alpha}, x, \dot{x}] \tag{17}$$

The complete set of equations describing the dynamics of the inverted pendulum is as follows:

$$\begin{aligned}\dot{\mathbf{V}} &= \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_4 \end{bmatrix} \\ &= \begin{bmatrix} v_2 \\ \frac{3g}{2l}\sin(v_1) + \frac{3a}{2l}[u(t) - v_4]\cos(v_1) - c_1 v_2 - c_2(v_2)^2 sgn(v_2) \\ v_4 \\ a[u(t) - v_4] \end{bmatrix}\end{aligned} \tag{18}$$

The control function *u(t)* is defined by the formulas (14-15). Please note the assumption that once the linear control is switched on, it remains on even if the absolute value of the pendulum angle $\alpha(t)$ happens to exceed α_0 again.

The simulation has been created using Python programming language with NumPy and SciPy packages. Integration of the set of equations (18) has been performed by means of the explicit Runge-Kutta method with step size control, implemented in the SciPy module. The following values of parameters have been used for the simulation: g = 9.81, l = 1.0, a =19.72688, c_1 = 0.07, c_2 = 0.01545, $u_{\max}$ = 2.0, $x_{\max}$ = 0.5, k_1 = 8.268, k_2 = 2.043, $k_3 = -$ 1.035, $k_4 = -$ 2.611. All these numbers have been identified on a real device. Values of the remaining parameters (t_s, δ, α_0) have been optimized in the simulation process by means of the Differential Evolution optimization algorithm [6] implemented in the SciPy module. The optimization target was to find values of the parameters (t_s, δ, α_0) for which the swing-up time t_f is the shortest. Precisely, the simulation has been stopped as soon as the length of the vector $\mathbf{V}$ has become smaller than 10^{-2}. The boundaries for parameters optimization have been chosen as follows: [0.0, 2.0] for t_s, [0.0, 2.0] for δ and [0.0, 0.5] for α_0.

The system defined by the set of equations (18) has been executed 2749 times by the optimization procedure in the search for the parameters (t_s, δ, α_0) that result in the shortest regulation time. The optimal values of parameters are as follows: $t_s = 0.94136$, $\delta = 1.23408$, $\alpha_0 = 0.43160$. The shortest swing-up time that corresponds to these parameters is equal to $t_f = 5.69800$.

The graph describing motion of the pendulum's bar in the optimized swing-up is presented in the Fig. 2. Motion of the cart and the control signal for the optimal swing-up parameters are presented in the Fig. 3.

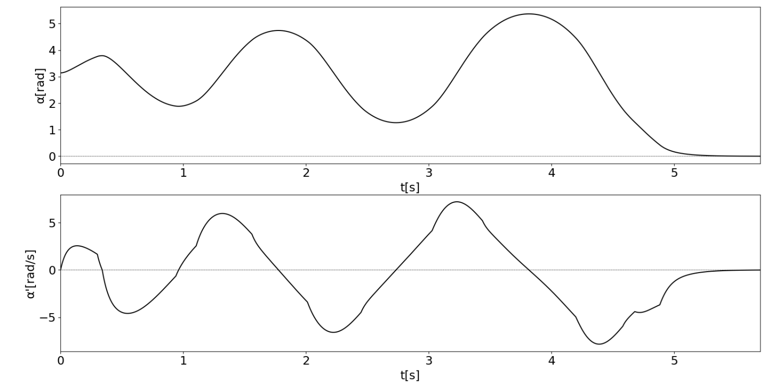

Figure 2 Motion of the pendulum's bar in the optimal swing-up – the angle and the angular velocity

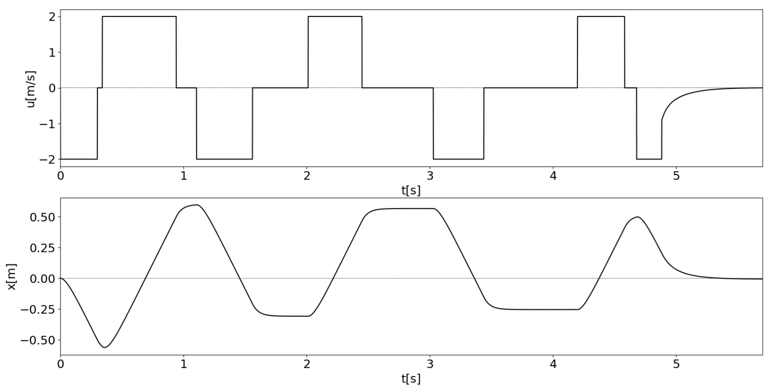

Figure 3 The control signal and motion of the cart in the optimal swing-up

4. Conclusions

This paper presents a novel approach to the swing-up problem of the inverted pendulum. It has been shown that the energy-space method can be applied to determine an efficient swing-up algorithm. The description of the problem in terms of energy vectors has been provided. On the basis of energy space considerations, a swing-up control has been proposed. The optimal control theory states that in order to minimize the time needed to drive the inverted pendulum from one state to another, only extreme control signal values should be used. Therefore, the algorithm is based on the "bang-bang" control. In order to improve the quality of regulation, a fixed starting time and a dead-zone for signal switching have been introduced. The linear control has been applied to stabilize the pendulum in the vicinity of the upright vertical position. Parameters of the swing-up algorithm have been optimized by means of the Differential Evolution method. The proposed swing-up algorithm is simple and results in fast transition of the inverted pendulum from the initial state to the desired final state.

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