Mechanics and Mechanical Engineering
Vol. 22, No. 3 (2018) 809–819

https://doi.org/10.2478/mme-2018-0063

Displacements Caused by the Temperature in Multicomponent, Multi-Layered Periodic Material Structures

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Received (19 July 2018)
Revised (21 August 2018)
Accepted (25 October 2018)

The present study aims to analyse a two-dimensional problem of displacements in theory of thermal stresses for multicomponent, multi-layered periodic composites. The model equations are obtained within the framework of the tolerance modelling procedure. These equations allow to determine the distribution of displacements caused by the temperature field in the theory of thermal stresses. The paper presents an example of a solution of a boundary value problem.

Keywords: theory of thermal stresses, periodic multicomponent composite, tolerance modelling.

1. Introduction

The object of the presented study is tolerance modelling for thermal stresses in a multicomponent, multi-layered periodical structure. Tolerance modelling (the tolerance averaging technique) for two-component structures was proposed by Woźniak [12, 13]. Applications of this theory in thermomechanics can be found in monographies by Woźniak and Wierzbicki [14], Woźniak, Michalak and Jędrysiak [15] as well as Woźniak et al. [16] and for example in paper by Wierzbicki, Woźniak and Woźniak [8]. Tolerance and asymptotic models of thermo-elasticity problems for two-component transversally graded laminates were proposed by Jędrysiak [2] as well as Pazera and Jędrysiak [5]. A similar problem was analysed by Ostrowski [4] for a two-component longitudinally graded hollow cylinder. Thermal stresses in periodic two-component multi-layered structures were considered for example by

Bagdasaryan [1] and using the theory of microlocal parameters by Matysiak [3]. The basic difference between the modelling of multicomponent composites and the modelling of two-component composites is the new form of the shape function which is called an oscillating micro-shape function. The idea of modelling for multi-layered, multicomponent composites was developed by Woźniak [10, 11]. It was used for periodic structures, e.g. for problems of heat conduction, by Wągrowska and Szlachetka [6] and for elastostatics problems by Wągrowska, Szlachetka and Bagdasaryan [7]. The aim of this paper is to present model equations for a two-dimensional problem of thermal stresses in temperature and displacements using the forms of oscillating micro-shape functions introduced in [6], [7], and to present certain solutions of boundary problems.

2. Object of analysis

The object of analysis is a periodic, multicomponent, multi-layered, thermoelastic composite which occupies a region $\Omega \equiv (0,L_1) \times (0,L_2) \times (0,L_3)$ in the physical space and consists of a large number N $\left(\frac{1}{N} << 1\right)$ of layers with constant thickness η, $\eta = \frac{L_1}{N}$.

Each layer is composed of P sublayers made of M homogeneous, orthotropic, perfectly combined linear thermoelastic materials, where $P \geq M$. Let us assume that the axes of orthotropy of the components coincide with the axes of the coordinate system $Ox_1x_2x_3$. Figure 1 presents the scheme of the considered composite.

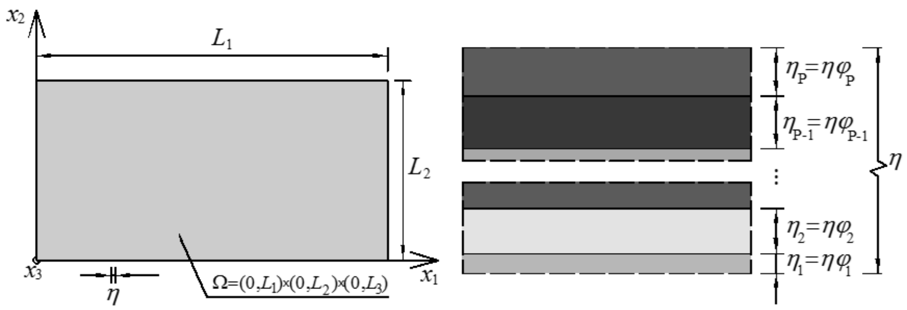

Figure 1 The scheme of a multicomponent multi-layered periodic composite and its periodic layer

For the two-dimensional problem the thermoelastic material properties in the p-th, $p = 1, ..., P$, orthotropic sublayer are described by the values of the elastic modulus, heat conduction and thermal expansion tensors:

$$C_p = \begin{bmatrix} C_p^{1111} & C_p^{1122} & 0 \\ & C_p^{2222} & 0 \\ & & C_p^{1212} \end{bmatrix} \qquad K_p = \begin{bmatrix} K_p^{11} & 0 \\ 0 & K_p^{22} \end{bmatrix}$$

$$D_p = \begin{bmatrix} D_p^{11} & 0 \\ 0 & D_p^{22} \end{bmatrix} \qquad p = 1, .., P$$

Moreover, let φ_p, $p = 1, ..., P$, are the fractional functions, such that $\varphi_1 + ... + \varphi_P = 1$. The thickness of the p-th ($p = 1, .., P$) sublayer in each layer is equal to $\eta_p = \eta\varphi_p$. Introduce the decomposition of the i-th interval of periodicity into P subintervals Δ_p^i which are defined as:

$$\Delta_p^i \equiv \left(\eta\,(i-1) + \sum_{k=1}^{p-1} \varphi_k \eta_k,\ \eta\,(i-1) + \sum_{k=1}^{p} \varphi_k \eta_k \right), p = 1,2, ..., P, i = 1,2, ..., N$$

Then the set occupied by the p-th sublayers in this composite can be described as follows:

$$\Omega_p = \bigcup_{i=1}^{N} \Delta_p^i \times (0, L_2) \times (0, L_3)\,, p = 1,2, ..., P$$

For the stationary, two-dimensional problems given within the framework of the thermal stresses theory, the temperature and the displacements depend only on two variables x_1 and x_2, where $x_1 \in (0, L_1)$, $x_2 \in (0, L_2)$.
Assuming that the body forces per unit volume are equal to zero, the equations of equilibrium for a two-dimensional problem take the form:

$$\left(C^{ijkl} u_{k,l}\right)_{,j} = \left(G^{ij}\theta\right)_{,j} \tag{1}$$

$$\left(K^{kl}\theta_{,l}\right)_{,k} = 0 \tag{2}$$

where θ is the temperature field, u_i are components of the displacement field and $G^{ij} = C^{ijkl} D^{kl}$, $i, j, k, l = 1, 2$. These are the equations which allow to determine the displacements field caused by the temperature field.
For orthotropic, homogeneous components of the composite above equations take the form:

$$C^{1111} u_{1,11} + \left(C^{1122} + C^{1212}\right) u_{2,12} + C^{1212} u_{1,22} = G^{11}\theta_{,1} \tag{3}$$

$$C^{1212} u_{2,11} + \left(C^{1122} + C^{1212}\right) u_{1,12} + C^{2222} u_{2,22} = G^{22}\theta_{,2} \tag{4}$$

$$K^{11}\theta_{,11} + K^{22}\theta_{,22} = 0 \tag{5}$$

where:
$C^{ijkl} = C_p^{ijkl}$, $G^{kl} = G_p^{kl}$ and $K^{kl} = K_p^{kl}$ when $(x_1, x_2) \in \bigcup_{i=1}^{N} \Delta_p^i \times (0, L_2)$, $p = 1,2, ..., P$, $i = 1,2, ..., N$.
Equations (3-5) are a system of partial differential equations with discontinuous and highly oscillating coefficients. There are many methods of finding the approximated solution of these equations. Among them the following methods can be distinguished: asymptotic homogenization, modelling with microlocal parameters and tolerance modelling. This paper applies the tolerance modelling method.

3. Modelling concepts

In the process of tolerance modelling for periodic composites notions of an averaging operator, slowly varying function (SV), and tolerance averaging approximation are needed. They can be found for example in [16] and [7]. The averaging operator for $f \in L^2((0, L_1))$ is defined by:

$$\langle f \rangle (x) \equiv \frac{1}{\eta} \int_{x-\eta/2}^{x+\eta/2} f(z)\, dz, x \in \left(\frac{\eta}{2}, L_1 - \frac{\eta}{2} \right) \tag{6}$$

Let denote an arbitrary convex set in the space R^m as Π and an arbitrary real-valued function $f \in C^1(\Pi)$. Define the tolerance parameter $d \equiv (\eta, \delta_0, \delta_1)$ as a triplet of real positive numbers. The notation $\partial_j \equiv \frac{\partial}{\partial x_j}$, $j = 1, ..., m$ will be used.

Function $f \in C^1(\Pi)$ is slowly varying function with respect to parameter d $(f \in \mathrm{SV}_d^1(\Pi)) \Leftrightarrow (\forall \ (\mathbf{x}, \mathbf{y}) \in \Pi^2 \, \|\mathbf{x} - \mathbf{y}\| \leq \eta \Rightarrow (|f(\mathbf{x}) - f(\mathbf{y})| \leq \delta_0 \wedge |\partial_j f(\mathbf{x}) - \partial_j f(\mathbf{y})| \leq \delta_1) \wedge (\forall \mathbf{x} \in \Pi \ \eta \, |\partial_j f(\mathbf{x})| \leq \delta_0), \ \forall j = 1, ..., m)$

If $f \in L^2\left(-\frac{\eta}{2}, \frac{\eta}{2}\right)$ and $F \in \mathrm{SV}_d^1((0, L_1))$ then the tolerance averaging approximation of $\langle fF \rangle_T(x)$, $\langle f \partial_1 F \rangle_T(x)$ is given by$\langle f \rangle (x) F(x)$ and $\langle f \rangle (x) \partial_1 F(x)$, respectively.

The main role in the modelling process for temperature and displacements in thermal stresses theory for multicomponent composites is played by oscillating micro-shape functions, which are defined separately for the heat conduction problem (function $\gamma^\theta(\cdot)$) and the elasticity problem (function $\gamma^u(\cdot)$). These functions are piecewise linear, with values on the interfaces between sublayers of a periodicity layer given by:

$$\gamma_p^\theta = \gamma_{p-1}^\theta + \eta \varphi_p \left(\frac{K_0^{11}}{K_p^{11}} - 1 \right), \ \ p = 1,2, ..., P$$

where $K_0^{11} \equiv \left(\sum_{i=1}^P \frac{\varphi_i}{K_i^{11}} \right)^{-1}$ and $\langle \gamma^\theta \rangle = 0$

$$\gamma_p^u = \gamma_{p-1}^u + \eta \varphi_p \left(\frac{C_0^{1111}}{C_p^{1111}} - 1 \right), \ \ p = 1,2, ..., P$$

where $C_0^{1111} \equiv \left(\sum_{i=1}^P \frac{\varphi_i}{C_i^{1111}} \right)^{-1}$ and $\langle \gamma^u \rangle = 0$

A scheme of a general oscillating micro-shape function γ (which could be γ^θ or γ^u) for a symmetric, three-component structure is presented in Fig. 2.

4. Modelling procedure and modelling equations for a two-dimensional problem

The process of tolerance modelling is based on two assumptions. The first assumption says that the temperature field $\theta(\cdot)$ and the displacement field $\mathbf{u}(\cdot)$ are approximated by $\tilde{\theta}(\cdot)$ and $\tilde{\mathbf{u}}(\cdot)$ respectively in the form, [16]:

$$\theta(x_1, x_2) \approx \tilde{\theta}(x_1, x_2) = \vartheta(x_1, x_2) + \gamma^\theta(x_1)\, \psi(x_1, x_2) \tag{7}$$

$$\mathbf{u}(x_1, x_2) \approx \tilde{\mathbf{u}}(x_1, x_2) = \mathbf{w}(x_1, x_2) + \gamma^u(x_1)\, \mathbf{v}(x_1, x_2) \tag{8}$$

Fields $\vartheta(\cdot, x_2)$, $\psi(\cdot, x_2)$, $\mathbf{w}(\cdot, x_2)$, $\mathbf{v}(\cdot, x_2) \in SV_d^1((0, L_1))$ are unknown fields, which are called macro-temperature, the amplitude of fluctuation of temperature, macro-displacement and the amplitude of fluctuation of displacement respectively.

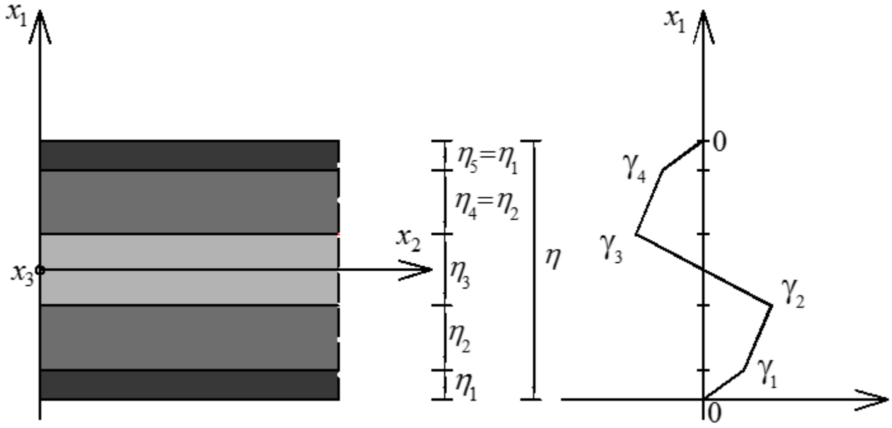

Figure 2 The scheme of an oscillating micro-shape function for a three-component periodic composite

Before formulating the second assumption let us define the residual fields of $\tilde{\theta}(\cdot)$ and $\tilde{\mathbf{u}}(\cdot)$ given by (7) and (8), [16]:

$$r^{\theta} = \left(K^{kl}\left(\vartheta + \gamma^{\theta}\psi\right),_{l}\right),_{k} \tag{9}$$

$$r_i^u = \left(C^{ijkl}\frac{1}{2}\left((w_k+\gamma^u v_k),_l + (w_l+\gamma^u v_l),_k\right)\right),_j - \left(G^{ij}(\vartheta+\gamma^{\theta}\psi)\right),_j \tag{10}$$

where $i,j,k,l = 1,\ 2.$

The second assumption can be written with formulas:

$$\left\langle r^{\theta}\right\rangle_T = 0, \left\langle \gamma^{\theta}\, r^{\theta}\right\rangle_T = 0 \tag{11}$$

$$\langle r_i^u\rangle_T = 0, \langle \gamma^u\, r_i^u\rangle_T = 0 \tag{12}$$

where $i = 1,\ 2.$
From the conditions (11) and Eqs. (7) and (9) the equations for $\vartheta(\cdot)$ and $\psi(\cdot)$ are received:

$$\begin{aligned}
&\left\langle\left(K^{kl}\left(\vartheta+\gamma^{\theta}\psi\right),_l\right),_k\right\rangle_T = \left\langle\left(K^{kl}\vartheta,_l\right),_k\right\rangle_T + \left\langle\left(K^{kl}\gamma^{\theta},_l\psi\right),_k\right\rangle_T \\
&+\left\langle\left(K^{kl}\gamma^{\theta}\psi,_l\right),_k\right\rangle_T = 0 \\
&\left\langle\gamma^{\theta}\left(K^{kl}\left(\vartheta+\gamma^{\theta}\psi\right),_l\right),_k\right\rangle_T = \left\langle\gamma^{\theta}\left(K^{kl}\vartheta,_l\right),_k\right\rangle_T + \left\langle\gamma^{\theta}\left(K^{kl}\gamma^{\theta},_l\psi\right),_k\right\rangle_T \\
&+\left\langle\gamma^{\theta}\left(K^{kl}\gamma^{\theta}\psi,_l\right),_k\right\rangle_T = 0
\end{aligned}$$

Remembering the definition of tolerance averaging approximation and the fact that $\left\langle\gamma^{\theta}\right\rangle = 0,\ \left\langle K^{kl}\gamma^{\theta}\right\rangle = 0$ and that $\vartheta(\cdot,x_2),\ \psi(\cdot,x_2) \in SV_d^1((0,L_1))$it can be proved that:

$$
\begin{aligned}
&\left\langle \left(K^{kl}\vartheta_{,l}\right)_{,k}\right\rangle_T = \left\langle K^{kl}\right\rangle \vartheta_{,lk} \qquad \left\langle \left(K^{kl}\gamma^{\theta}{}_{,l}\,\psi\right)_{,k}\right\rangle_T = \left\langle K^{kl}\gamma^{\theta}{}_{,l}\right\rangle \psi_k \\
&\left\langle \left(K^{kl}\gamma^{\theta}\psi_{,l}\right)_{,k}\right\rangle_T = \left\langle \left(K^{kl}\gamma^{\theta}\right)\psi_{,lk}\right\rangle = 0 \\
&\left\langle \gamma^{\theta}\left(K^{kl}\vartheta_{,l}\right)_{,k}\right\rangle_T = \left\langle \left(\gamma^{\theta}K^{kl}\vartheta_{,l}\right)_{,k}\right\rangle_T - \left\langle \gamma^{\theta}{}_{,k}K^{kl}\vartheta_{,l}\right\rangle_T = -\left\langle \gamma^{\theta}{}_{,k}K^{kl}\right\rangle \vartheta_{,l} \\
&\left\langle \gamma^{\theta}\left(K^{kl}\gamma^{\theta}{}_{,l}\,\psi\right)_{,k}\right\rangle_T = \left\langle \left(\gamma^{\theta}K^{kl}\gamma^{\theta}{}_{,l}\,\psi\right)_{,k}\right\rangle_T - \left\langle \left(\gamma^{\theta}{}_{,k}K^{kl}\gamma^{\theta}{}_{,l}\,\psi\right)\right\rangle_T \\
&= -\left\langle \gamma^{\theta}{}_{,k}K^{kl}\gamma^{\theta}{}_{,l}\right\rangle \psi \\
&\left\langle \gamma^{\theta}\left(K^{kl}\gamma^{\theta}\psi_{,l}\right)_{,k}\right\rangle_T = \left\langle \left(\gamma^{\theta}K^{kl}\gamma^{\theta}\psi_{,l}\right)_{,k}\right\rangle_T - \left\langle \gamma^{\theta}{}_{,k}K^{kl}\gamma^{\theta}\psi_{,l}\right\rangle_T \\
&= \left\langle \left(\left(\gamma^{\theta}\right)^2 K^{kl}\right)\right\rangle \psi_{,kl}
\end{aligned}
$$

This procedure yields the equations, [15]:

$$\left\langle K^{11}\right\rangle \vartheta_{,11} + \left\langle K^{22}\right\rangle \vartheta_{,22} + \left\langle K^{11}\gamma^{\theta}{}_{,1}\right\rangle \psi_{,1} = 0 \tag{13}$$

$$\underline{\left\langle K^{22}\left(\gamma^{\theta}\right)^2\right\rangle}\psi_{,22} - \left\langle K^{11}\left(\gamma^{\theta}{}_{,1}\right)^2\right\rangle \psi - \left\langle K^{11}\gamma^{\theta}{}_{,1}\right\rangle \vartheta_{,1} = 0 \tag{14}$$

Similarly, inserting Eqs. (8) and (10) to Eqs. (12) and considering Eq. (6) and the fact that $\langle\gamma^u\rangle = 0$, $\left\langle C^{ijkl}\gamma^u\right\rangle = 0$and that $w\left(\cdot,x_2\right),\ v\left(\cdot,x_2\right) \in SV_d^1\left((0,L_1)\right)$ the second group of model equations take the form, [15]:

$$
\begin{aligned}
&\left\langle C^{1111}\right\rangle w_{1,11} + \left\langle C^{1111}\gamma^u_{,1}\right\rangle v_{1,1} + \left\langle C^{1122}\right\rangle w_{2,21} + \left\langle C^{1212}\right\rangle w_{2,12} \\
&+\left\langle C^{1212}\gamma^u{}_{,1}\right\rangle v_{2,2} + \left\langle C^{1212}\right\rangle w_{1,22} = \left\langle G^{11}\right\rangle \vartheta_{,1}
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&-\left\langle C^{1111}\gamma^u{}_{,1}\right\rangle w_{1,1} - \left\langle C^{1111}\left(\gamma^u{}_{,1}\right)^2\right\rangle v_1 + \underline{\left\langle C^{1111}\left(\gamma^u\right)^2\right\rangle}v_{1,11} \\
&-\left\langle C^{1122}\gamma^u{}_{,1}\right\rangle w_{2,2} + \underline{\left\langle C^{1122}\left(\gamma^u\right)^2\right\rangle}v_{2,21} + \underline{\left\langle C^{1212}\left(\gamma^u\right)^2\right\rangle}v_{2,12} \\
&+\underline{\left\langle C^{1212}\left(\gamma^u\right)^2\right\rangle}v_{1,22} = -\left\langle G^{11}\gamma^{\theta}{}_{,1}\right\rangle \vartheta + \underline{\left\langle G^{11}\left(\gamma^{\theta}\right)^2\right\rangle}\psi_{,1}
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&\left\langle C^{1212}\right\rangle w_{2,11} + \left\langle C^{1212}\gamma^u{}_{,1}\right\rangle v_{2,1} + \left\langle C^{1212}\right\rangle w_{1,21} + \left\langle C^{1122}\right\rangle w_{1,12} \\
&+\left\langle C^{1122}\gamma^u{}_{,1}\right\rangle v_{1,2} + \left\langle C^{2222}\right\rangle w_{2,22} = \left\langle G^{22}\right\rangle \vartheta_{,2}
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
&-\left\langle C^{1212}\gamma^u{}_{,1}\right\rangle w_{2,1} - \left\langle C^{1212}\left(\gamma^u{}_{,1}\right)^2\right\rangle v_2 + \underline{\left\langle C^{1212}\left(\gamma^u\right)^2\right\rangle}v_{2,11} \\
&-\left\langle C^{1212}\gamma^u{}_{,1}\right\rangle w_{1,2} + \underline{\left\langle C^{1212}\left(\gamma^u\right)^2\right\rangle}v_{1,21} + \underline{\left\langle C^{1122}\left(\gamma^u\right)^2\right\rangle}v_{1,12} \\
&+\underline{\left\langle C^{2222}\left(\gamma^u\right)^2\right\rangle}v_{2,22} = \underline{\left\langle G^{22}\left(\gamma^{\theta}\right)^2\right\rangle}\psi_{,2}
\end{aligned} \tag{18}
$$

The system of partial differential Eqs. (13) – (18) with (7) and (8) with boundary conditions for the temperature and the displacements represent what will be called the standard tolerance model for a thermal stresses problem.

It should be emphasized that the system of Eqs. (13) – (18) obtained in the process of tolerance modelling has constant coefficients in contrast to the Eqs. (3) – (5).

The underlined components in Eqs. (14), (16) and (18) depend on the length parameter η. If $\eta \to 0$, then these components vanish. From Eq. (14) it follows that $\psi = -\left\langle K^{11}\gamma^{\theta}{}_{,1}\right\rangle \vartheta_{,1}\left\langle K^{11}\left(\gamma^{\theta}{}_{,1}\right)^2\right\rangle^{-1}$ and from Eqs. (16) and (18) that:

$$v_1 = \frac{\left\langle G^{11}\gamma^{\theta}{}_{,1}\right\rangle \vartheta - \left\langle C^{1111}\gamma^{u}{}_{,1}\right\rangle w_{1,1} + \left\langle C^{1122}\gamma^{u}{}_{,1}\right\rangle w_{2,2}}{\left\langle C^{1111}\left(\gamma^{u}_{,1}\right)^2\right\rangle}$$

$$v_2 = -\frac{\left\langle C^{1212}\gamma^{u}{}_{,1}\right\rangle \left(w_{1,2} + w_{2,1}\right)}{\left\langle C^{1212}\left(\gamma^{u}{}_{,1}\right)^2\right\rangle}$$

In view of above determined amplitudes of fluctuation of temerature (ψ) and displacements (v_1 and v_2), Eqs. (13), (15) and (17) take the form:

$$K_0^{11}\vartheta_{,11} + \left\langle K^{22}\right\rangle \vartheta_{,22} = 0 \tag{19}$$

$$C_0^{1111}w_{1,11} + \left(\tilde{C}^{1122} + C_0^{1212}\right) w_{2,12} + C_0^{1212}w_{1,22} = G_0^{11}\vartheta_{,1} \tag{20}$$

$$C_0^{1212}w_{2,11} + \left(C_0^{1212} + \tilde{C}^{1122}\right) w_{1,12} + \tilde{C}^{2222}w_{2,22} = G_0^{22}\vartheta_{,2} \tag{21}$$

with constant coefficients:

$$K_0^{11} = \left\langle K^{11}\right\rangle - \frac{\left(\left\langle K^{11}\gamma^{\theta}{}_{,1}\right\rangle\right)^2}{\left\langle K^{11}\left(\gamma^{\theta}{}_{,1}\right)^2\right\rangle}$$

$$C_0^{1111} = \left\langle C^{1111}\right\rangle - \frac{\left(\left\langle C^{1111}\gamma^{u}{}_{,1}\right\rangle\right)^2}{\left\langle C^{1111}\left(\gamma^{u}{}_{,1}\right)^2\right\rangle}$$

$$\tilde{C}^{1122} = \left\langle C^{1122}\right\rangle - \frac{\left\langle C^{1111}\gamma^{u}{}_{,1}\right\rangle\left\langle C^{1122}\gamma^{u}{}_{,1}\right\rangle}{\left\langle C^{1111}\left(\gamma^{u}{}_{,1}\right)^2\right\rangle}$$

$$C_0^{1212} = \left\langle C^{1212}\right\rangle - \frac{\left(\left\langle C^{1212}\gamma^{u}{}_{,1}\right\rangle\right)^2}{\left\langle C^{1212}\left(\gamma^{u}{}_{,1}\right)^2\right\rangle}$$

$$\tilde{C}^{2222} = \left\langle C^{2222}\right\rangle - \frac{\left(\left\langle C^{1122}\gamma^{u}{}_{,1}\right\rangle\right)^2}{\left\langle C^{1111}\left(\gamma^{u}{}_{,1}\right)^2\right\rangle}$$

$$G_0^{11} = \left\langle G^{11}\right\rangle - \frac{\left\langle C^{1111}\gamma^{u}{}_{,1}\right\rangle\left\langle G^{11}\gamma^{\theta}{}_{,1}\right\rangle}{\left\langle C^{1111}\left(\gamma^{u}{}_{,1}\right)^2\right\rangle}$$

$$G_0^{22} = \left\langle G^{22}\right\rangle - \frac{\left\langle C^{1122}\gamma^{u}{}_{,1}\right\rangle\left\langle G^{22}\gamma^{\theta}{}_{,1}\right\rangle}{\left\langle C^{1111}\left(\gamma^{u}{}_{,1}\right)^2\right\rangle}$$

5. Example

In this section the distribution of an approximate displacement field caused by given temperature load for multicomponent multi-layered periodic composite is presented. Unknown temperature and displacements are calculated within the framework of the theory of thermal stresses using Eqs. (19) – (21). It is assumed that all materials of the discussed composites are homogeneous and isotropic, so the values of heat and elastic modules are reduced to: $K^{11} = K^{22} = K, C^{1111} = C^{2222} = 2\mu+\lambda$, $C^{1122} = \lambda$, $C^{1212} = \mu$, $G^{11} = G^{22} = G = (3\lambda + 2\mu)\,\alpha_t$ where K is the coefficient of heat conduction, λ, μ are Lamé parameters and α_t is the coefficient of thermal expansion. Let us assume that the composite occupies the region $\Omega \equiv (0,L_1) \times (0,L_2)$, where $L_1 = 1.2$ [m], $L_2 = 1$ [m] and is composed of $N = 12$ layers with constant thicknesses $\eta = 0.1$ [m]. It means that the thickness of the periodicity layer is equal to 0.1 [m]. The periodicity layer consists of five sublayers made of three different materials. Thicknesses of sublayers "1", "5", are equal to $0,1\,\eta$, thicknesses of sublayers "2", "4" are equal to $0,2\,\eta$ and the thickness of sublayer "3" is equal to $0,4\,\eta$. The sublayers made of the same material are distributed symmetrically with respect to the midplane of the periodicity layer. The parameters related to the corresponding sublayers in the considered case are shown in Tab. 1.

The graphs of the oscillating micro-shape functions $\gamma^{\theta}(\cdot)$ and $\gamma^{u}(\cdot)$ in the periodicity layer for the considered example are shown in Fig. 3. It should be noted that if sublayers made of the same material are symmetrically distributed with respect to the midplane of the periodicity layer, the graph of the oscillating micro-shape function is antisymmetric with respect to this midplane and that the oscillating micro-shape function is equal to zero on the edges of the periodicity layer.

Table 1 The parameters related to the corresponding sublayers in the considered case

Sublayer	1	2	3	4	5
K [W/mK]	35	200	380	200	35
λ ($\cdot 10^{10}$) [Pa]	4.583	5.108	9.515	5.108	4.583
μ ($\cdot 10^{10}$) [Pa]	0.625	2.632	4.478	2.632	0.625
α_t ($\cdot 10^{-5}$) [1/K]	2.9	2.3	1.65	2.3	2.9

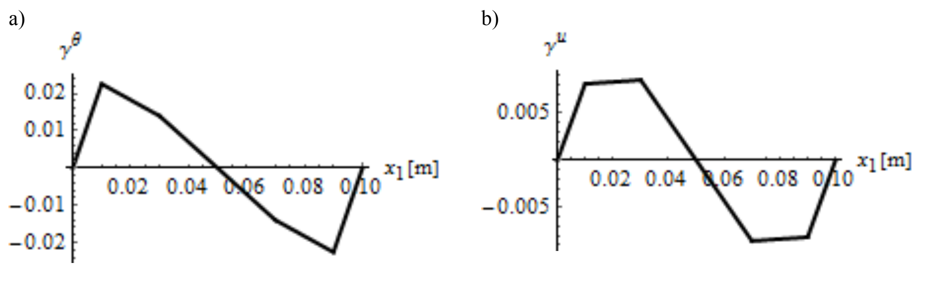

Figure 3 Graphs of the oscillating micro-shape functions in periodicity layer a) $\gamma^{\theta}(\cdot)$, b) $\gamma^{u}(\cdot)$

The boundary conditions for the displacements in the presented case are:
$\vartheta(0,x_2) = f_1(x_2)$, $\vartheta(x_1,0) = f_2(x_1)$, $\vartheta(L_1,x_2) = 0$, $\vartheta(x_1,L_2) = 0$, $w_1(0,x_2) = 0$, $w_1(L_1,x_2) = 0$, $w_1(x_1,0) = 0$, $w_1(x_1,L_2) = 0$ and $w_2(0,x_2) = 0$, $w_2(L_1,x_2) = 0$, $w_2(x_1,0) = 0$, $w_2(x_1,L_2) = 0$, where $f_1(x_2) = \vartheta_0 \sin\left(\frac{\pi x_2}{L_2}\right)$, $f_2(x_1) = \vartheta_0 \sin\left(\frac{\pi x_1}{L_1}\right)$, $\vartheta_0 = 400\,[\mathrm{K}]$.

The distributions of the macro-temperature ϑ and the temperature $\tilde{\theta}$ are shown in Fig. 4. The macro-displacements w_1 and w_2, as well as displacements $\tilde{u}_1$ and $\tilde{u}_2$ are shown in Fig. 5 and Fig. 6, respectively. Figure 7 presents the cross-sections of the macro-displacements w_1 and w_2, as well as displacements $\tilde{u}_1$and $\tilde{u}_2$ for $x_2 = 0.1L_2$, $x_2 = 0.25L_2$ and $x_2 = 0.5L_2$.

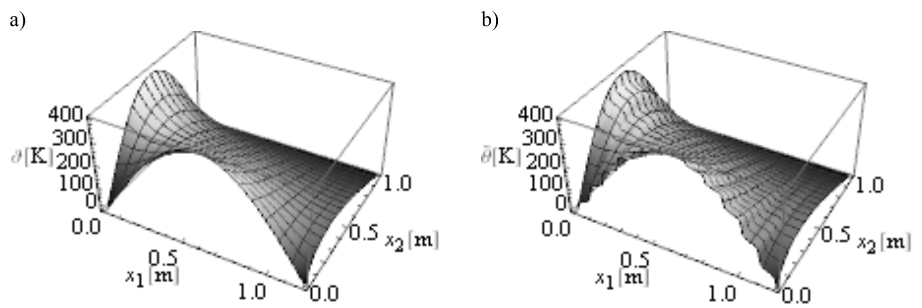

Figure 4 The distributions of a) macro-temperature ϑ, and b) approximated temperature $\tilde{\theta}$

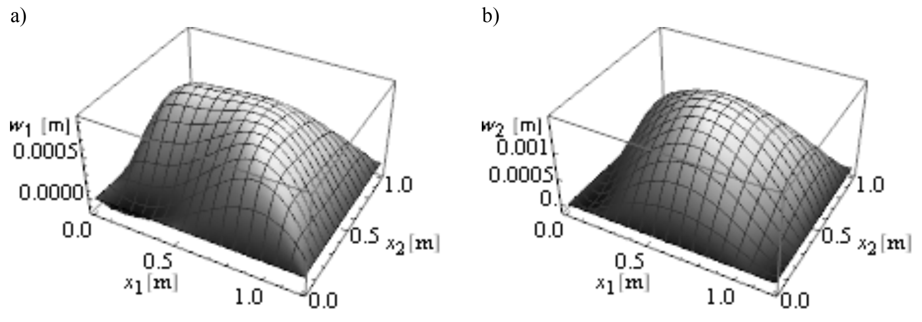

Figure 5 The distributions of the macro-displacements a) w_1 , b) w_2

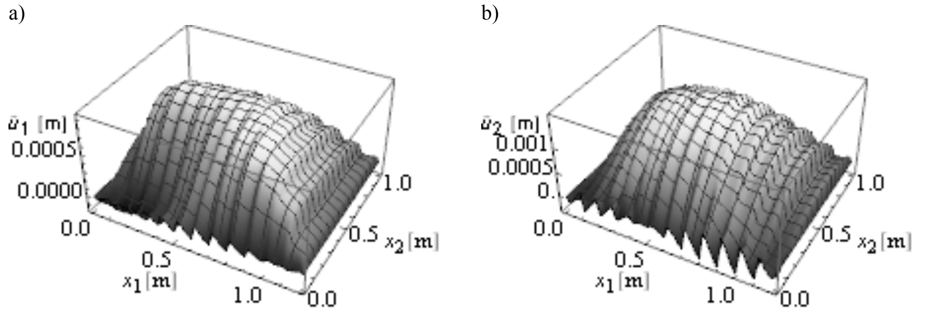

Figure 6 The distributions of the approximated displacements a) $\tilde{u}_1$, b) $\tilde{u}_2$

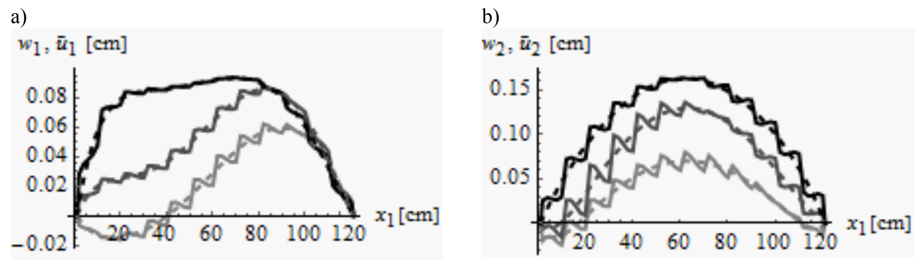

Figure 7 The distributions of the approximated displacements $\tilde{u}_1$ and $\tilde{u}_2$ (the continuous line) and macro-displacements w_1 and w_2 (the dashed line) for $x_2 = 0.1L_2$ – the light grey line, $x_2 = 0.25L_2$ – the dark grey line, $x_2 = 0.5L_2$ – the black line

6. Conclusions

Introducing oscillating micro-shape functions give the possibility to modelling for multicomponent, multi-layered periodic composites. These functions do not depend only on the composite's geometry but also on the material properties of individual components. This fact gives the odds of allow to pass form a non-homogeneous structure to a homogeneous one. Presented model equations can be also used for two-component structures. It has to be emphesised that for thermoelasticity problems two different oscillating micro-shape functions must be used – for temperature and for displacements.

In the presented paper the distribution of temperature and displacement were received. The obtained model equations gives the possibility to determine the stresse from the known constitutive relations for theory of thermal stresses.

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