Mechanics and Mechanical Engineering
Vol. 22, No. 3 (2018) 719–726

https://doi.org/10.2478/mme-2018-0056

Fourier Variant Homogenization of the Heat Transfer Processes in Periodic Composites

Ewaryst Wierzbicki
Dorota Kula
Łukasz Wodzyński
Departments of Civil Engineering
Faculty of Civil and Enviromental Engineering
Warsaw University of Life Sciences
Warsaw, Poland
el_el@o2.pl
kuladorota@wp.pl
lukaszwodzynski@op.pl

Received (1 July 2018)
Revised (10 August 2018)
Accepted (18 October 2018)

The well-known parabolic Heat Transfer Equation is a simplest recognized description of phenomena related to the heat conductivity in solids with microstructure. However, it is a tool difficult to use due to the discontinuity of coefficients appearing here. The purpose of the paper is to reformulate this equation to the form that allows to represent solutions in the form of Fourier's expansions. This equivalent re-formulation has the form of infinite number of equations with Fourier coefficients in expansion of the temperature field as the basic unknowns. The first term in Fourier representation, being an average temperature field, should satisfy the well-known parabolic heat conduction equation with Fourier coefficients as fields controlling average temperature behavior. The proposed description takes into account changes of the composite periodicity accompanying changes in the variable perpendicular to the surfaces separating components, concerning *FGM*-type materials and can be treated as the asymptotic version of Heat Transfer Equation obtained as a result of a certain limit passage where the cell size remains unchanged.

Keywords: temperature fluctuations, homogenization, tolerance modelling.

1. Introduction

The starting point for the implementation of the proposed method of modeling is the modeling method known as *the Tolerance Averaging Technique* (*TAT*). It was proposed by Professor Czesław Woźniak. The reader is referred to the six basic monograph on this subject [1-6]. In this paper instead of Woźniak's *Micro-Macro Hypothesis* (used in tolerance modeling) the more general hypothesis has been applied. It introduce possibility of developing a Fourier series for residuals

between the exact temperature and its micro-macro hypothetical approximation. Hence, in the proposed course of modeling the temperature field is represented as a sum of two parts. The first part, referred to as *the long-wave part of the temperature field*, coincides with the temperature mentioned in the tolerance micro-macro hypothesis. The second part is the *short-wave part of the temperature field* represented by the oscillating terms of the properly selected Fourier expansion.
The starting point of considerations is the well-known parabolic heat transfer equation:

$$\nabla^T(K\nabla\theta) - c\dot{\theta} = b \tag{1}$$

in which the region $\Omega \subset R^D$, $2 \leq D \leq 3$, occupied by the composite is restricted to:

$$\Omega = \Omega_\sigma \times \Omega_{D-\sigma} \tag{2}$$

where:
1^o $\Omega_\sigma = (0, L)$, $\Omega_{D-\sigma} = (0, \delta_1) \times (0, \delta_2)$ while $(\sigma, D) = (1, 3)$,
2^o $\Omega_\sigma = (0, L_1) \times (0, L_2)$, $\Omega_{D-\sigma} = (0, \delta)$ while $(\sigma, D) = (2, 3)$,
3^o $\Omega_\sigma = (0, L)$, $\Omega_{D-\sigma} = (0, \delta)$ while $(\sigma, D) = (1, 2)$
for $L_1, L_2, L, \delta_1, \delta_2, \delta > 0$. In (1) $\theta = \theta(y, z, t)$, $y \in \Omega_\sigma \subset R^\sigma$, $z \in \Omega_{D-\sigma} \subset R^{D-\sigma}$, $t \geq 0$, denotes the temperature field, c is a specific heat field and k is the heat conductivity constant. Moreover, $\nabla \equiv \nabla_\sigma + \nabla_{D-\sigma}$ for $\nabla_\sigma \equiv [\partial/\partial y^1, ..., \partial/\partial y^\sigma, 0, .., 0]^T$ with zeros placed in $D - \sigma$ positions and $\nabla_{D-\sigma} \equiv [0, ..., 0, \partial/\partial z^1, ..., \partial/\partial z^{D-\sigma}]^T$ with zeros placed in σ positions. Fields $c = c(\cdot)$ and $k = k(\cdot)$ take$S$ values denoted by $c^1, \ldots, c^S$ and $k^1, \ldots, k^S$, respectively, and do not depend on the temperature field θ and are restrictions to Ω_σ of a certain periodic fields defined in R^σ. The paper is restricted to the microstructural Δ-periodic composites which diameter $\lambda = diam(\Delta)$ not necessary small where compared to the characteristic length dimension of the region Ω. It means that there exists σ-tuple $(\mathbf{v^1}, ..., \mathbf{v}^\sigma)$ of independent vectors $(\mathbf{v^1}, ..., \mathbf{v}^\sigma)$ from R^σ, determining m directions of periodicity and referred to as periodicity vectors, such that region:

$$\bigcup\{\Delta_{k_1,...k_\sigma,r_{\sigma+1},...,r_3} : \quad k_1, ...k_\sigma = ... - 1, 0, 1, ..., \quad r_{\sigma+1}, ..., r_3 \in R\} = R^3 \tag{3}$$

for $\Delta_{k_1,...k_\sigma,r_{\sigma+1},...,r_3} \equiv k_1 v^1 + ... + k_\sigma v^\sigma + r_{\sigma+1} e^{\sigma+1} + ... + r_3 e^3 + \Delta$ and such that both fields $c = c(\cdot)$ and $K = K(\cdot)$ are Δ- periodic. Here e^i denotes i-th unit vector from R^3, $i = 1, 2, 3$. In the subsequent investigations an important role plays the averaging $\langle f\rangle(x)$, $z = z(z, y)$, of an arbitrary integrable field$f$defined on R^σdefined as region:

$$\langle f\rangle(x) = \frac{1}{|\Delta|}\int_\Delta f(\xi)d\xi \tag{4}$$

and which is a constant field provided that $f$is a Δ-periodic field.

2. Modelling procedure

The investigations is based on the two fundamental assumptions. *The first modeling assumption* is a certain extension of *the micro-macro hypothesis* introduced framework of *the tolerance averaging technique*, cf. [1-6]. In accordance with that hypothesis, the temperature field θ can be approximated with an acceptable accuracy with form:

$$\theta_M(z) = \vartheta(z) + h^A(x)\psi_A(z) \tag{5}$$

In which the slowly varying fields $\vartheta(\cdot)$ and $\psi_A(\cdot)$ are *tolerance averaging of temperature field* and *amplitude fluctuations fields*, respectively. Here and in the sequel the summation convention holds with respect to indices . $A = 1, ..., N$ Symbols h^A, $A = 1, ..., N$, denote in (5) tolerance shape functions which should be periodic and satisfy conditions:

$$h^A \in o(\lambda), \quad \lambda\nabla_y h^A \in o(\lambda), \quad \langle ch^A\rangle = 0 \tag{6}$$

For particulars the reader is referred to [1-6]. We suggests to interpret in (5) $\theta_{long} = \vartheta$ and $\theta_{short} = h^A(x)\psi_A(z)$. The *tolerance-micro macro hypothesis* can be formulated in the form:

Tolerance micro-macro hypothesis. The residual part of the temperature field θ_{res} being the difference between the temperature field θ and its tolerance parts θ_M given by (5) can be treated as zero, $\theta_{res} \equiv \theta - \theta_M \approx 0$, i.e. it vanish with an acceptable "tolerance approximation".

In contrast to the tolerance modeling in this paper instead of quoted above micro - macro hypothesis *the extended micro - macro hypothesis* will be applied. According to this new hypothesis the right side of (6) is not equal to zero but is a special infinite analytic expansion. As for the attempt to adapt the idea implemented in the application of the theory of signals, where we have to deal with the "overlapping" of many signals determining by different parameters. In order to formulate this hypothesis denote by $\Delta_1, \Delta_2, ..., \Delta_n$the homogeneity subregions of the basic cell Δ and let $\Gamma_1 = \partial\Delta_1$,..., $\Gamma_n = \partial\Delta_n$, $\Gamma_\Delta = \Gamma_1 \cup ... \cup \Gamma_n$. Now, *the extended micro - macro hypothesis* can be formulated as follow:

Assumption 1. (Extended micro-macro hypothesis)
The smoothed part $\theta_{reg} \equiv \theta - \theta_{res}$ of the temperature field θ being the difference between temperature field $\theta$and its residual part θ_{res} given by (5) produces disappearing heat flux vector $q_{reg} \equiv k\nabla\theta_{reg} = 0$ in regular points of the composite discontinuity surfaces.

In the subsequent considerations it will be assumed that summation convention holds also with respect to $p = 1, 2, ...$ That is why, under the second modeling assumption, formula (6) can be rewritten in the form:

$$\theta(y, z, t) - \theta_{res}(y, z, t) = a_0(z, t) + a_p(z, t)\phi^p(y, z) \tag{7}$$

In according to *the second modeling assumption* shape functions as well as the orthogonal system $\phi^p(x)$ are independent on the thermal and geometrical properties of the conductor.

Remark 1. Hence, short-wave part $\theta_{reg} \equiv \theta - \theta_{res}$ of the temperature field can be represented by Fourier expansion:

$$\theta_{reg}(x, y, t) = a_0(z, t) + \phi^p a_p(z, t) \tag{8}$$

formed by an orthogonal Δ - periodic basis $\phi^p(x)$, $p = 1, 2, ...$ If orthogonality is here related to the scalar product $f_1 \circ f_2 = \langle f_1 f_2\rangle = \sum_{s=1}^{S} \eta_s \langle f_1 f_2\rangle_s$, determined by

the averaged values $\langle f_1 f_2\rangle_s$ taken over homogeneous parts Δ_s of the repetitive cell $\Delta \subset R^\sigma$, then Fourier basis can be treated as independent on the material structure of the composite.
Remark 2. The residual temperature field θ_{res} can be represented in the tolerance form $\theta_{res} = \langle\theta_{res}\rangle + h^A\psi_A$ for a certain oscillating shape functions $h^A = h^A(y,z,t)$, $\langle h^A\rangle = 0$, and amplitudes $\psi_A = \psi_A(y,t)$. Hence θ_{res} can be treated as a micro-macro part (given by (5) and (6)) of the temperature field θ. That is why expansion:

$$\theta = \vartheta + \lambda[g^A\psi_A + a_p(z,t)\varphi^p(y,z)] \tag{9}$$

is a certain temperature representation formed for:

$$\vartheta = a_0 + \langle\theta_{res}\rangle \tag{10}$$

and for $h^A(x,t) \equiv \lambda g^A(\lambda^{-1}x)$ and $\phi^p(x,t) \equiv \lambda\varphi^p(\lambda^{-1}x)$ together with the conditions

$$\begin{array}{l} \langle c\varphi^p\rangle = 0, \quad \langle K\varphi^p\rangle = 0, \quad p = 1,2,\dots\,, \\ \langle ch\rangle = 0, \quad \langle Kh\rangle = 0. \end{array} \tag{11}$$

formulated under *Remark 1.*
Remark 3. Assumption 1. ensures fulfillment the well-known continuity condition imposed on the heat flux component $(q)_n \equiv n^T K\nabla\theta$ in the normal direction to the composite discontinuity surfaces Γ (determined by continuous unit vector field $n = n(\cdot)$ normal to these surfaces) provided that residual heat flux $(q_{res})_n \equiv n^T K\nabla(\theta_{res})$ is also continuous.

Now we are to formulate *third modeling assumption* crucial for formulation of the final reformulation of parabolic heat transfer equation. It is and additional assumption which provides the ability to perform tolerance modeling procedure with respect to the fields $u = \langle\vartheta\rangle$ as the average temperature field, and to the fields $\psi(\cdot)$and $a_p(\cdot)$ as fluctuation amplitudes, respectively.

Assumption 2. (*The locality of the long-wave part of the temperature field*)
The temperature field in the ε- neighborhood of the surface separating the components. Smoothing operation leading from θ to θ_{reg} can be realized under additional condition $\theta_{res} \neq 0$ if $x \in \Gamma_\varepsilon$, and hence $g = g_\varepsilon \to 0$ and $\vartheta = \vartheta_\varepsilon \to u$ while $\varepsilon \to 0$.

Under *Assumption 3.* we can treat condition $\vartheta = \vartheta_\varepsilon \to u$ as a special definition of the average temperature field.

3. Governing equations

Assumptions 1, 2, 3, yield the conclusion that if temperature field is represented by (7) then limit passage with $\varepsilon \to 0$ leads from the heat transfer equation to model equations:

$$\begin{array}{l} \langle c\dot{u}\rangle - \nabla^T[\langle k\rangle\nabla u - \langle k\nabla^T\varphi^p\rangle a_p - \langle k\nabla g^A\rangle\psi_A] = -\langle b\rangle \\ \langle\nabla^T g^A k\nabla g^B\rangle\psi_B + \langle k\nabla^T g^A\rangle\nabla u = 0 \\ \lambda^2\{\langle\varphi^p c\varphi^q\rangle\dot{a}_q - \nabla_z^T\langle\varphi^p c\varphi^q\rangle\nabla_z a_q\} + \lambda(\langle\nabla_y^T\varphi^p k\varphi^q\rangle - \langle\nabla^T\varphi^q k\varphi^p\rangle)\nabla_z a_q + \\ +\langle\nabla_y^T\varphi^p k\nabla\varphi^q\rangle a_q + \langle\nabla_y^T\varphi^p k\nabla g^A\rangle\psi + \langle k\nabla_y^T\varphi^p\rangle\nabla u = \lambda\langle\phi^p b\rangle \end{array} \tag{12}$$

which will be referred to as *The Extended Tolerance Model of Heat Conduction in Periodic Composites* . In (12) tolerance shape functions g^A as well as all coefficients including these shape functions are treated as limit passages with $\varepsilon \to 0$ of g_ε^A and of the related averaged coefficients including tolerance shape functions g_ε^A, respectively. In fact equations (14) are related to the case in which periodic structure of the composite depends on z-variable i. e.. In this case of periodicity we use the name *FGM?type composite materials* to emphasize the importance of physical phenomena that become important when the periodic structure of the composite changes with the change of $z$variable. Equations (14) will be referred to as *The Extended Tolerance Model of Heat Conduction in FGM?type Periodic Composites.*

It should be emphasized that including the dependence of the composite periodicity on the z-variable averaged coefficients in Equations(12) become dependent on full gradients ∇ of the shape function in places before the right-hand bracket segment $\rangle$denoting the averaging over the repetitive cell Δ instead of the dependence on partial gradients ∇_y with respect to the periodic variable y in the same places, when we restrict ourselves to the saturation η that does not change along with the both variables y and z. That is why matrix coefficient $\lambda(\langle\nabla_y^T\varphi^p k\varphi^q\rangle - \langle\nabla^T\varphi^q k\varphi^p\rangle)$ in the damping term $\lambda(\langle\nabla_y^T\varphi^p k\varphi^q\rangle - \langle\nabla^T\varphi^q k\varphi^p\rangle)\nabla_z a_q$ can be decomposed on the sum onto not-vanishing symmetric and skew-symmetric terms. Situations in which *FGM*?periodicity reduces to the usual periodicity the symmetric term in this decomposition vanish. In the asymptotic homogenization approach, cf. [1,2], the dependence the composite periodicity on the "z" variable is possible to consider by the including to the original homogenized equations the special correctors that allow the fulfillment of appropriate boundary conditions impossible to fulfillment during their absence.

4. Passage to the Effective Conductivity

Eliminations of Fourier ψ_A and tolerance amplitudes a_p (if such eliminations are possible) lead to the single equation for average temperature field u named as *Effective Conductivity Equation.* It is easy to verify that from (12) $\psi_A$can be formally eliminated and hence *the Extended Tolerance Model* can be written as:

$$\begin{aligned}
&\langle c\dot{u}\rangle - \nabla^T(k^\perp\nabla u + \langle k\nabla^T\varphi^p\rangle a_p) = -\langle b\rangle \\
&\lambda^2(\langle\varphi^p c\varphi^q\rangle\dot{a}_q - \nabla_z^T\langle\varphi^p c\varphi^q\rangle\nabla_z a_q) + \lambda(\langle\nabla^T\varphi^p k\varphi^q\rangle - \langle\nabla^T\varphi^q k\varphi^p\rangle)\nabla_z a_q + \\
&+\langle\nabla_y^T\varphi^q k\nabla\varphi^p\rangle a_p + \langle\nabla_y^T\varphi^p k\nabla g^A\rangle\psi + [k]^\perp\nabla u = \lambda\langle\varphi^p b\rangle
\end{aligned} \tag{13}$$

with:

$$\begin{aligned}
k^\perp &= \langle k\rangle - \langle k\nabla g^A\rangle\langle\nabla^T g^A k\nabla g^B\rangle^{-1}\langle\nabla^T g^B k\rangle \\
[k]^\perp &= \langle k\nabla_y^T\varphi^p\rangle - \langle k\nabla_y^T\varphi^p\nabla g^A\rangle\langle\nabla^T g^A k\nabla g^B\rangle^{-1}\langle\nabla^T g^B k\rangle
\end{aligned} \tag{14}$$

used as the projection of the *Effective Conductivity Constant* onto the z–variable direction. The investigation of the reduction of Fourier amplitudes a_p from (13) leading to the *Effective Conductivity Constant* projection onto the y–variable direction is still an open problem and was carried out only in special cases of one-directional periodicity.

5. Governing equations in the periodic case

In periodic case equations (13) takes the form:

$$\begin{array}{l}\langle c\dot{u}\rangle - \nabla^T(k^{\perp}\nabla u + \langle k\nabla_y^T\varphi^p\rangle a_p) = -\langle b\rangle \\ \langle\nabla^T g^A k\nabla g^B\rangle\psi_B + \langle k\nabla^T g^A\rangle\nabla_z u = 0 \\ \lambda^2(\langle\varphi^p c\varphi^q\rangle\dot{a}_q - \nabla_z^T\langle\varphi^p c\varphi^q\rangle\nabla_z a_q) + \lambda(\langle\nabla_y^T\varphi^p k\varphi^q\rangle - \langle\nabla_y^T\varphi^q k\varphi^p\rangle)\nabla_z a_q + \\ +\langle\nabla_y^T\varphi^p k\nabla\varphi^q\rangle a_q + \langle k\nabla_y^T\varphi^p\rangle\nabla_z u = \lambda\langle\varphi^p b\rangle\end{array} \tag{15}$$

in which coefficients are obtained from the related coefficients in (14) by replacement of full gradients of ∇g^A, $\nabla\varphi^q$ as well as of its transposed counterparts $\nabla^T\varphi^p$, $\nabla^T\varphi^p$ in the interior of the averaged coefficients in (14) are changed by $\nabla_y g^A$, $\nabla_y\varphi^q$, $\nabla_y^T\varphi^p$, $\nabla^T\varphi^p$, respectively. It must be emphasized that coefficient $\langle\nabla_y^T\varphi^p k\varphi^q\rangle - \langle\nabla_y^T\varphi^q k\varphi^p\rangle$ in the damping term in (13) creates skew-symmetric $\sigma\times\sigma$ matrix in contrast to the corresponding coefficient $\langle\nabla_y^T\varphi^p k\varphi^q\rangle - \langle\nabla^T\varphi^q k\varphi^p\rangle$ in (15) creating matrix that is generally not antysymmetric.

6. Two-phased one-directional periodicity

Impulses:

$$\begin{array}{l} f_p(y) = \left\{\begin{array}{ll}\frac{\lambda}{2} - \alpha_p[1+\cos 2\pi p(\frac{y}{\eta\lambda}+1)] & dla \quad -\eta\lambda \le y \le 0 \\ \frac{\lambda}{2} - \alpha_p[1+\cos 2\pi p(\frac{\bar{y}}{\eta\lambda}+1)]|_{\bar{z}=0} & dla \quad 0 \le y \le (1-\eta)\lambda\end{array}\right. \\ f_{p+m}(y) = \left\{\begin{array}{ll}\frac{\lambda}{2} - \alpha_{p+m}[1+\cos 2\pi p(\frac{\bar{y}z}{(1-\eta)\lambda}-1)]|_{\bar{z}=0} & dla \quad -\eta\lambda \le y \le 0 \\ \frac{\lambda}{2} - \alpha_{p+m}[1+\cos 2\pi p(\frac{y}{(1-\eta)\lambda}-1)] & dla \quad 0 \le y \le (1-\eta)\lambda\end{array}\right. \\ f_{p+2m}(y) = \left\{\begin{array}{ll} -\frac{\lambda}{2}\cos(2p-1)\pi(\frac{y}{\eta\lambda}+1) & dla \quad -\eta\lambda \le y \le 0 \\ -\frac{\lambda}{2}\cos(2p-1)\pi(\frac{y}{(1-\eta)\lambda}-1) & dla \quad 0 \le y \le (1-\eta)\lambda\end{array}\right.\end{array} \tag{16}$$

where:

$$\alpha_p = \frac{\lambda}{2}\bar{\alpha} \equiv \frac{\lambda}{2(1+\eta)}, \quad \alpha_{p+m} = \frac{\lambda}{2}\underline{\alpha} \equiv \frac{\lambda}{2(2-\eta)} \tag{17}$$

subjected orthogonality procedure realized by:

$$\begin{array}{l}\lambda\varphi_p = f_p + \alpha f_{p+m}, \\ \lambda\varphi_{p+m} = f_p - \alpha f_{p+m}, \\ \lambda\varphi_{p+2m} = f_{p+2m}.\end{array} \tag{18}$$

for the independent on p and m parameter:

$$\alpha = \pm\frac{\langle kf_p f_p\rangle}{\langle kf_{p+m}f_{p+m}\rangle} \tag{19}$$

and treated as Fourier basis mentioned in *Assumption* 1 for the case of two-phased one-directionally periodic composite ensures fulfillment all *Assumptions* 1.2.3. That is why equations can be treated as an equivalent reformulation of heat transfer equation for periodic composites *s.*

7. Final Remarks

The Extended Tolerance Model is an alternative representation of heat transfer equation for periodic composites. The infinite number of equations in (13) is an important inconvenience of both limit model and the extended tolerance model. Although, there are many possibilities to avoid this inconvenience. It must be emphasized that the means of basic unknowns ψ_A, a_p before and after completing the limit passage $\varepsilon \to +\infty$ can be different. Particularly, terms $g^A(x)\psi(x,y)$ in (11) vanish and hence representation of temperature field given by extended micro-macro hypothesis changes to the form follow:

$$\theta(y,z,t) = u(z,t) + \lambda a_p(z,t)\varphi^p(y,z) \tag{20}$$

cannot be conclude that temperature gradient $\nabla\theta(x,y,t)$ is equal to $\nabla\theta(x,y,t) = \nabla u(z,t) + \nabla[\lambda a_p(z,t)\varphi^p(y,z)]$ since from (5) one can obtain:

$$\begin{array}{l} \nabla_y\theta(y,z) = \lim_{\varepsilon\to 0}\{\nabla_y u(z) + \nabla_y[g^A_\varepsilon(y,z)\psi_A(z) + \varphi^p_\varepsilon(y,z)a_p(z)\} \\ \nabla_z\theta(y,z) = \lim_{\varepsilon\to 0}\{\nabla_z u(z) + \nabla_z[g^A_\varepsilon(y,z)\psi_A(z) + \varphi^p_\varepsilon(y,z)a_p(z)\}]\} \\ \frac{d}{dt}\theta(y,z) = \lim_{\varepsilon\to 0}\{\frac{du}{dt}(z) + \frac{d}{dt}[g^A_\varepsilon(y,z)\psi_A(z) + \varphi^p_\varepsilon(y,z)a_p(z)\} \end{array} \tag{21}$$

as a certain representations of the related derivatives of the temperature field. The model equations obtained in this study are based on [3,9,11] and are an extension of the equations received for the rigid composite periodicity. Papers [12,13] include applications of these equations. model. Papers [8,10] can be treated as such applications in the special rigid composite periodicity case.

Equations obtained in the paper constitute the realization of a certain conviction of Cz. Woźniak that the asymptotic version of tolerance model (obtained as a result of the limit passage in which the size of the repetitive cell tends to zero and which allows the determination of the the tensor of effective modules) should be obtained as a result of the another limit passage in which the cell size remains unchanged. Model equations in the paper have been obtained on this way. Mentioned new parameter tending to zero is the width of the ribbon surrounding the discontinuity surfaces including the support of the used tolerance shape functions.

References

[1] **Bensoussan, A., Lions, J.-L. and Papanicolaou, G.**: Asymptotic Analysis for Periodic Structures, *American Math. Soc.*, ISBN-10: 0-8218-5324-4, ISBN-13: 978-0-8218-5324-5, **2011**.

[2] **Ariault, J.L.**: Effective macroscopic description for heat conduction in periodic composites, *International Journal of Heat and Mass Transfer*, 26, 6, 861-869, DOI: 10.1016/S0017-9310(83)80110-0, **1983**.

[3] **Woźniak, Cz. and Wierzbicki, E.**: *Averaging techniques In thermomechanics of composite solids, Tolerance averaging versus homogenization*, Technical University of Częstochowa Press, Częstochowa, **2000**.

[4] **Woźniak, C.** (**Ed**.): Thermomechanics of microheterogeneous solids and structures. Tolerance averaging approach, *Technical University of Łódź Press*, Łódź **2009**.

[5] **Woźniak, C.** (**Ed**.): Developments In Mathematical Modeling and Analysis of Microstructured Media, *Silesian University Press*, Gliwice **2010**.

[6] **Jędrysiak, J.**: Termomechanics of laminates, plates and shells with functionally graded properties /in Polish/, *Technical University of Łódź Press*, Łódź **2010**.

[7] **Michalak, B.**: Termomechanics of solids with a certain nonhonmogeneous microstructure: tolerance approximation technique /in Polish/, *Technical University of Łódź Press*, Łódź, **2010**.

[8] **Kula, D.**: Assessment of the impact of the geometric structure of periodic composites on the intensity of damping the fluctuation of boundary loads /PhD dissertation in Polish/, *Faculty of Civil Engineering*, *Architecture and Environmental Engineering*, *Łódź University of Technology*, **2016**.

[9] **Kula, D. and Wierzbicki, E.:** On the Fourier series implementation issue tolerance modeling thermal conductivity of periodic composites, *Engineering Transaction*, 63, 1, 77–92, **2015**.

[10] **Kula, D.:** On the existence of the sinusoidal–type temperature fluctuations independently suppressed by the periodic two – phased conducting layer, *Acta Scientarum Polonorum, Ser, Arch.*, 63, 1, 77-92, **2015**.

[11] **Woźniak, M., Wierzbicki, E. and Woźniak, C.**: A macroscopic model of the diffusion and heat transfer processes in a periodically micro-stratified solid layer, *Acta Mechanica*, 157, 175-185, **2002**.

[12] **Kula, D., Wierzbicki, E., Witkowska-Dobrev, J. and Wodzyński, Ł.:** Fourier variant homogenization treatment of one impulse Boundary effect behaviour, *this issue*, **2018**.

[13] **Wodzyński, Ł., Kula, D. and Wierzbicki, E.:** Transport of even and odd temperature fluctuations across the chess-board type periodic composite, *this issue*, **2018**.