STABILITY ANALYSIS OF CRITICAL POINTS

1. Analyzed system

$$J_0\ddot{\varphi} = \left(Mgl + mg\frac{l}{2} \right)\sin\varphi - K\varphi - C\dot{\varphi} \qquad (1)$$

where:

m – mass of the rod [kg]

M – mass concentrated at the end [kg]

l – length of the rod [m]

J_0 – mass moment of inertia, $J_0 = M l^2 + 1/3 m l^2$

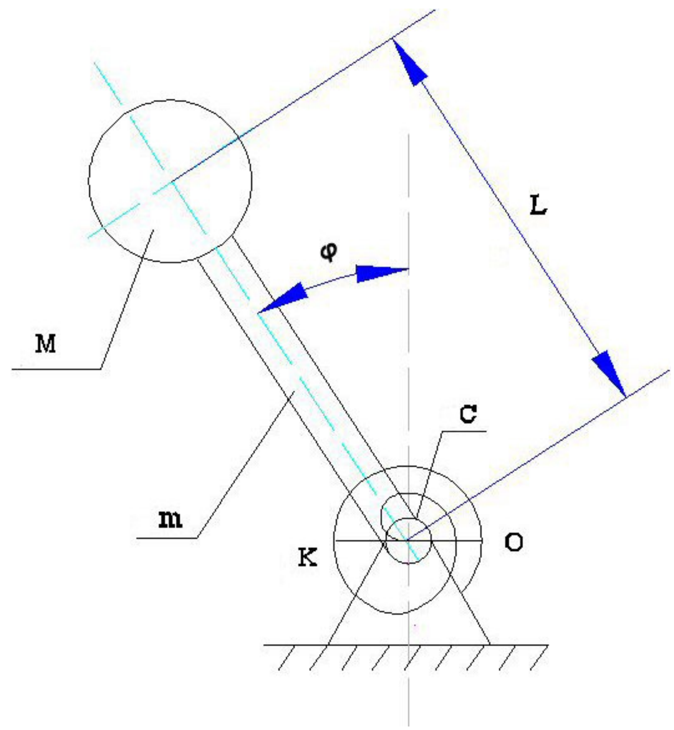
K – stiffness of spiral spring [Nm/rad]

C – equivalent viscous damping [Nms/rad]

φ - angular displacement [rad]

$\dot{\varphi}$ - angular velocity [rad/s]

$\ddot{\varphi}$ - angular acceleration [rad/s^2]



For $m \approx 0$ the system (1) is reduced to:

$$J_0 \ddot{\varphi} = Mgl \sin\varphi - K\varphi - C\dot{\varphi}, \tag{2}$$

Values of remaining system parameters:

M = 0.1 [kg], l = 0.19 [m], $J_0 = Ml^2$ = 0.0361 [kgm^2]

C, K – control parameters

2.1 Dimensionless equation of motion

The function $\sin\varphi$ in Taylor series:

$$\sin\varphi = \varphi - \frac{1}{3!}\varphi^3 + \frac{1}{5!}\varphi^5 - ...... \tag{3}$$

$$J_0 \frac{d^2\varphi}{dt^2} + C\frac{d\varphi}{dt} + K\varphi - Mgl\left(\varphi - \frac{1}{6}\varphi^3\right) = 0 \Big/ K \tag{4}$$

$$\frac{J_0}{K}\frac{d^2\varphi}{dt^2} + \frac{C}{K}\frac{d\varphi}{dt} + \frac{K}{K}\varphi - \frac{Mgl}{K}\left(\varphi - \frac{1}{6}\varphi^3\right) = 0 \tag{5}$$

Substituting $K = J_0\alpha^2$`, where $\alpha = \sqrt{\frac{K}{J_0}}$ we obtain

$$\frac{1}{\alpha^2}\frac{d^2\varphi}{dt^2} + \frac{C}{J_0\alpha^2}\frac{d\varphi}{dt} + \varphi - \frac{Mgl}{J_0\alpha^2}\left(\varphi - \frac{1}{6}\varphi^3\right) = 0. \tag{6}$$

Dimensionless time: $\tau = \alpha t \Rightarrow d\tau = \alpha dt$

$\frac{d\varphi}{d\tau} = \frac{1}{\alpha}\frac{d\varphi}{dt}$ – dimensionless angular velocity [-],

$\frac{d^2\varphi}{d\tau^2} = \frac{1}{\alpha^2}\frac{d\varphi}{dt}$ – dimensionless angular acceleration [-],

Hence,

$$\frac{d^2\varphi}{d\tau^2} + \frac{C}{J_0\alpha}\frac{d\varphi}{d\tau} + \varphi - \frac{Mgl}{J_0\alpha^2}\left(\varphi - \frac{1}{6}\varphi^3\right) = 0. \tag{7}$$

Dimensionless variables and parameters:

$$\varphi = y, \qquad \dot{y} = \frac{d\varphi}{d\tau}, \qquad \ddot{y} = \frac{d^2\varphi}{d\tau^2}, \qquad h = \frac{C}{J_0\alpha} = \sqrt{\frac{C^2}{J_0 K}}, \qquad q = \frac{Mgl}{J_0\alpha^2} = \frac{mgl}{K}$$

Final dimensionless equation of motion:

$$\ddot{y} + h\,\dot{y} + (1-q)y + \frac{1}{6}qy^3 = 0. \tag{8}$$

2.2 Equation of motion in the form of first order differential equations

Substitutions:

$y_1 = y$

$\dot{y}_1 = \dot{y} = y_2$

$\dot{y}_2 = \ddot{y}$

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -hy_2 - (1-q)y_1 - \frac{1}{6}qy_1^3 \end{aligned} \tag{9}$$

2.3. Critical points

Equation defining the position of critical points in the phase space:

$$\begin{cases} y_2 = 0 \\ -hy_2 - (1-q)y_1 - \frac{1}{6}qy_1^3 = 0 \end{cases} \tag{10}$$

The solution:

$$-(1-q)y_1 - \frac{1}{6}qy_1^3 = 0 \qquad \Longrightarrow \qquad y_1\left[(q-1) - \frac{1}{6}qy_1^2\right] = 0 \qquad \Longrightarrow$$

$$y_1 = 0 \qquad \text{or} \qquad (q-1) - \frac{1}{6}qy_1^2 = 0 \;\Longrightarrow\; y_1^2 = \frac{6(q-1)}{q}$$

$$y_{11} = 0 \;\vee\; y_{12} = \sqrt{\frac{6(q-1)}{q}} \;\vee\; y_{13} = -\sqrt{\frac{6(q-1)}{q}} = -y_{12}$$

$$y_{21} = y_{22} = y_{23} = y_2 = 0.$$

Critical points (equilibrium positions):

$$Y_1 = (y_{11}, y_2) = (0,0)$$

$$Y_{2,3} = (\pm y_{12}, y_2) = \left(\pm \sqrt{\frac{6(q-1)}{q}};0 \right)$$

For q<1 (i.e., $K > Mgl$) there exists only Y=(0,0).

Lagrange'a-Dirichlet criterion:

Potential energy: $V = \frac{1}{2} K\varphi^2 - Mgl(1-\cos\varphi)$

$$\frac{dV}{d\varphi} = K\varphi - Mgl\sin\varphi \quad \Longrightarrow \quad \frac{dV}{d\varphi} = 0. \ \text{ for } \ \varphi = 0$$

$$\frac{d^2V}{d\varphi^2} = K - Mgl\cos\varphi > 0 \quad \text{for } \ \varphi = 0 \quad \Longrightarrow \quad \boldsymbol{K > Mgl}$$

2.4 Linearization in the neighborhood of critical points

Dynamical system

$$\frac{dx}{dt} = f(x) \tag{11}$$

where $f(x)$ is a differentiable function and $x \in \mathbf{R}^n$. In the neighborhood of critical point $x = a$ we have:

$$\frac{dx}{dt} = \frac{\partial f(a)}{\partial x}(x-a) + \text{.......+ components of higher order} \tag{12}$$

Linearization:

$$\frac{dx}{dt} = \frac{\partial f(a)}{\partial x}(x-a), \tag{13}$$

where

$\partial f(a)/\partial x$ is a Jacobi matrix.

Substituting $y = x - a$, and $\partial f(a)/\partial x = A$ we obtain

$$\frac{dy}{dt} = Ay. \tag{14}$$

General form of Jacobi matrix for the analyzed system:

$$A = \begin{bmatrix} 0 & 1 \\ (q-1) - 3\frac{1}{6}qy_{1i}^2 & -h \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ (q-1) - \frac{1}{2}qy_{1i}^2 & -h \end{bmatrix} \tag{15}$$

$i = 1, 2, 3.$

Point Y_1

$$A = \begin{bmatrix} 0 & 1 \\ (q-1) - \frac{1}{2} q y_{11}^2 & -h \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ (q-1) & -h \end{bmatrix}. \qquad (16)$$

Linearized equations of motion:

$$\dot{y}_1 = y_2,$$
$$\dot{y}_2 = (q-1) y_1 - h y_2. \qquad 17)$$

Point $Y_{2,3}$

$$A = \begin{bmatrix} 0 & 1 \\ (q-1) - \frac{1}{2} q y_{12}^2 & -h \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2(q-1) & -h \end{bmatrix}. \qquad (18)$$

Linearized equations of motion:

$$\dot{y}_1 = y_2,$$

$$\dot{y}_2 = -2(q-1)y_1 - hy_2 \pm 2(q-1)\sqrt{\frac{6(q-1)}{q}}. \tag{19}$$

2.5 Eigenvalues of Jacobi matrix

<u>Point Y_1</u>

$$\det(A-\lambda \mathrm{I}) = \begin{vmatrix} 0 & 1 \\ (q-1) & -h \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 1 \\ (q-1) & -(h+\lambda) \end{vmatrix} = 0. \tag{20}$$

$$\lambda(\lambda+h) - (q-1) = 0, \quad \Longrightarrow \quad \lambda^2 + h\lambda - (q-1) = 0, \quad \Longrightarrow \quad \Delta = \sqrt{h^2 + 4(q-1)}$$

$$\Longrightarrow \quad \lambda_{1,2} = \frac{-h \pm \sqrt{h^2+4(q-1)}}{2} \tag{21}$$

Point $Y_{2,3}$

$$\det\left(A-\lambda \mathrm{I}\right)=\begin{vmatrix} -\lambda & 1 \\ -2(q-1) & -\left(h+\lambda\right) \end{vmatrix}=0, \tag{22}$$

$$\lambda^2+h\lambda+2(q-1)=0, \qquad \Longrightarrow \qquad \Delta=\sqrt{h^2-8(q-1)}$$

$$\lambda_{1,2}=\frac{-h\pm\sqrt{h^2-8(q-1)}}{2}. \tag{23}$$

2.6 Stability analysis of point Y_1

Consider two cases:

1) $C=0$

2) $C=\ 0{,}025\ \left[\frac{Nms}{rad}\right]$

Ad.1 (free undamped vibrations case)

$$C=0 \quad \Longrightarrow \quad h=0 \quad \Longrightarrow \quad \lambda_{1,2}=\frac{1}{2}\sqrt{4(q-1)}=\sqrt{q-1}$$

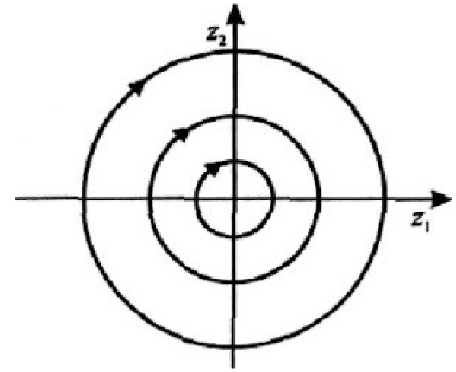


Fig.2

a) $q<1 \quad \Longrightarrow \quad Mgl < K \quad \Longrightarrow \quad \lambda_{1,2}=\pm i\sqrt{1-q}$

Two imaginary eigenvalues – critical point of centre type (Fig.2)

b) $q \geq 1 \Rightarrow Mgl > K$

$$\lambda_{1,2}=\pm\sqrt{q-1}$$

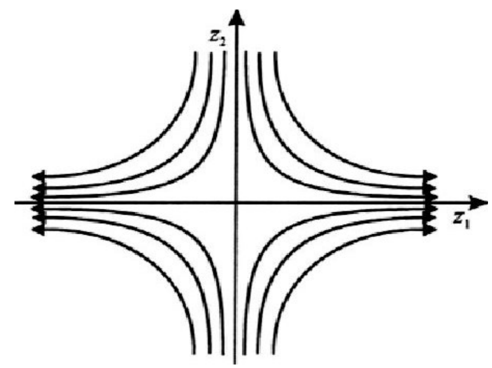


Fig.3

Two real eigenvalues of different signs – critical point of saddle type (Fig.3)

Ad. 2 (free damped vibrations case)

$$C = 0{,}025 \left[\frac{Nms}{rad}\right] \quad \Longrightarrow \quad h = \sqrt{\frac{C^2}{J_0 K}} \quad \Longrightarrow$$

$$\lambda_{1,2} = -\frac{1}{2}\sqrt{\frac{C^2}{J_0 K}} \pm \sqrt{\frac{C^2}{J_0 K} + 4\left(\frac{Mgl}{K} - 1\right)} \tag{24}$$

From (24) results:

a) for $\frac{C^2}{J_0 K} + 4\left(\frac{Mgl}{K} - 1\right) \geq 0,$ i.e. $K \leq \frac{C^2}{4J_0} - Mgl$, there exist two real eigenvalues of different signs – critical point of saddle type (Fig.3).

The boundary value of K_b for assumed system parameters is

$$K_b = \frac{(0{,}025)^2}{4 * 0{,}0361} + 0{,}1 * 9{,}81 * {,}019,$$ hence

$$K_b \leq 0{,}191 \text{ [Nm/rad]}$$

b) for $\dfrac{C^2}{J_0K}+4\left(\dfrac{Mgl}{K}-1\right)<0$, i.e. $K_b>0{,}191$ [Nm/rad] there exist Two complex eigenvalues of Jacobi matrix (Eq.(16))

$$\lambda_{1,2}=-\frac{1}{2}\sqrt{\frac{C^2}{J_0K}}\pm i\sqrt{\left|\frac{C^2}{J_0K}+4\left(\frac{Mgl}{K}-1\right)\right|}.$$

Due to negative real part the critical point is a stable focus now (Fig.4).

(a) (b)

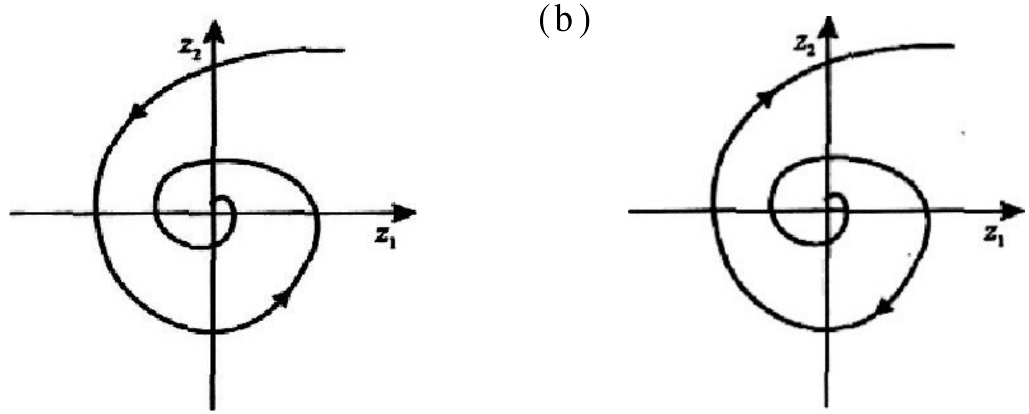


Fig.4

2.6 Stability analysis of points $Y_{2,3}$

Stiffness of spiral spring – $K = 0{,}17$ [Nm/rad] $\Longrightarrow$ $q = 1{,}1$

Position of critical points in phase space:

$$y_{1,2} = \sqrt{\frac{6(q-1)}{q}} = 0{,}54\, rad \approx 31^0$$

$$y_{1,3} = -\sqrt{\frac{6(q-1)}{q}} = -0{,}54\, rad \approx -31^0$$

a) Critical damping

Dimensionless: $h^2 - 8(q-1) = 0,$ $\Longrightarrow$ $h = \sqrt{8(q-1)} = 0{,}894$

real: $C = \sqrt{8J_0(Mgl - K)}$ $\Longrightarrow$ $C = 0{,}069\left[\frac{Nms}{rad}\right]$

b) Undercritical damping

$$h^2 < 8(q-1), \qquad \Longrightarrow \qquad h < 0,894$$

$$\lambda_{1,2} = -\frac{h}{2} \pm i\sqrt{8(q-1)-h^2}.$$

Two complex eigenvalues of Jacobi matrix (Eq.(18)) with negative real part – the critical point is a stable focus (Fig.4).

b) Overcritical damping

$$h^2 > 8(q-1), \qquad \Longrightarrow \qquad h > 0,894$$

$$\lambda_{1,2} = -\frac{h}{2} \pm \frac{1}{2}\sqrt{h^2 - 8(q-1)}.$$

Two real negative eigenvalues – critical point of node type

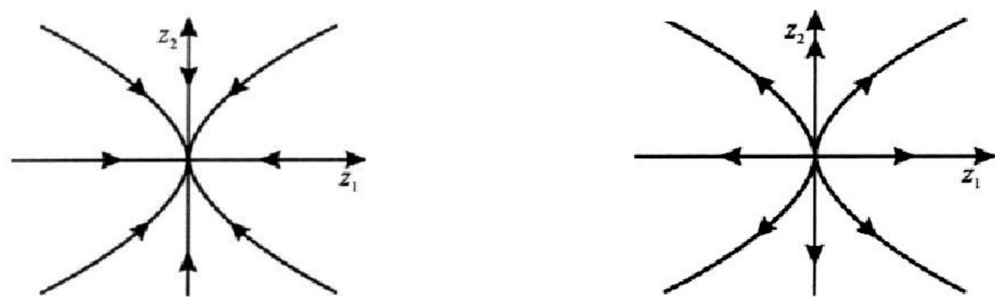


Fig.5

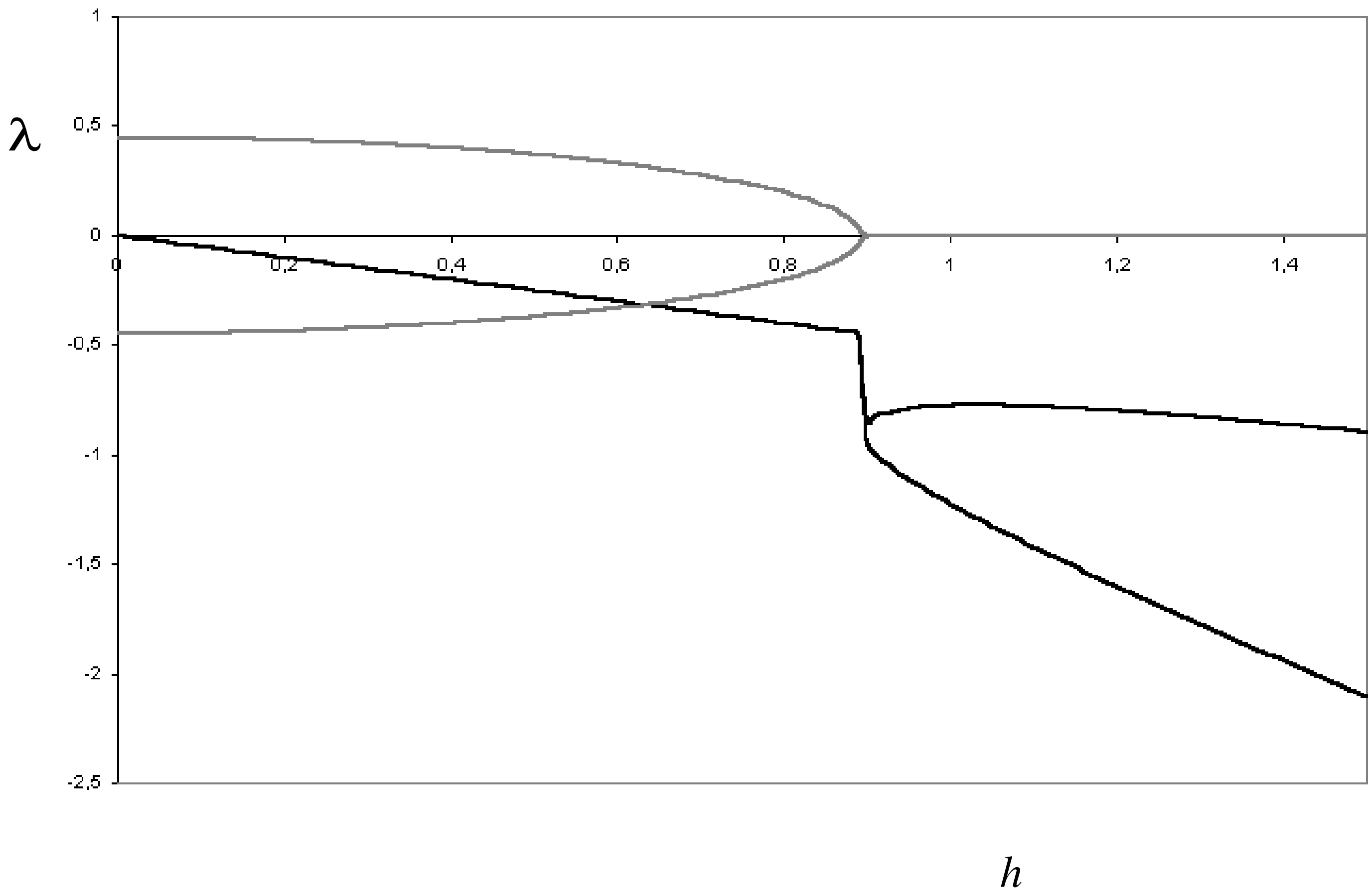


Fig.6

Basins of attraction

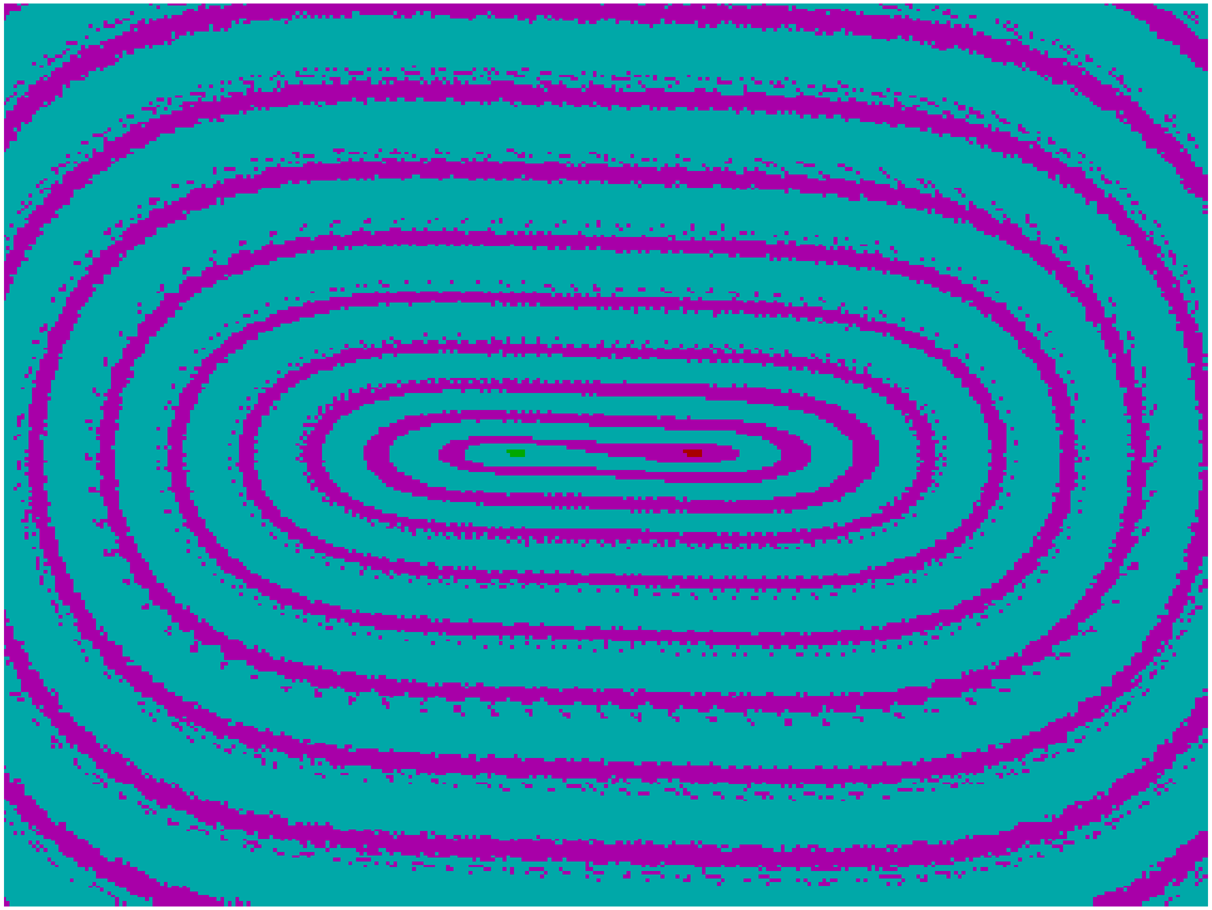
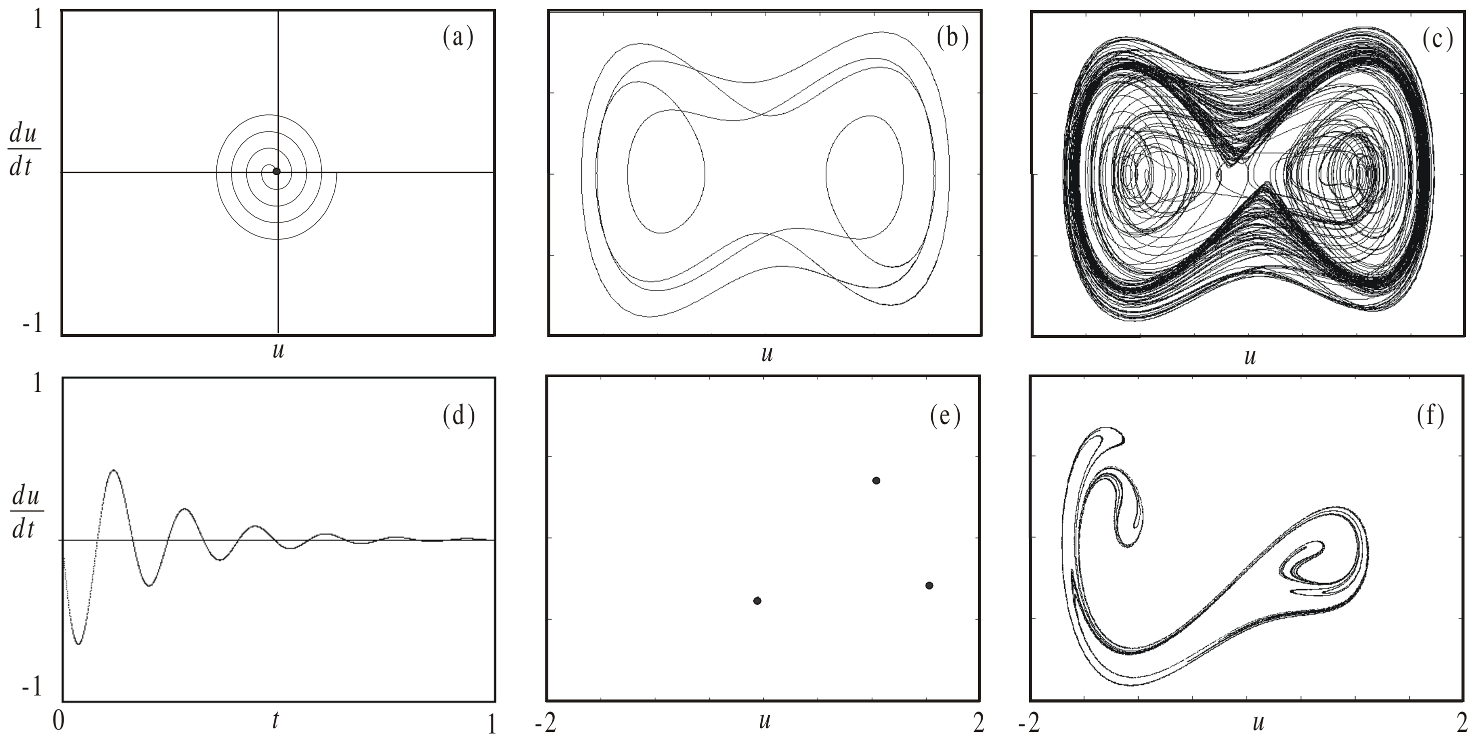
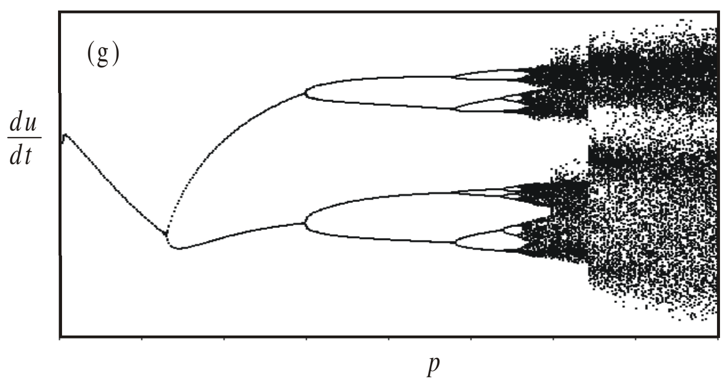


Fig.7



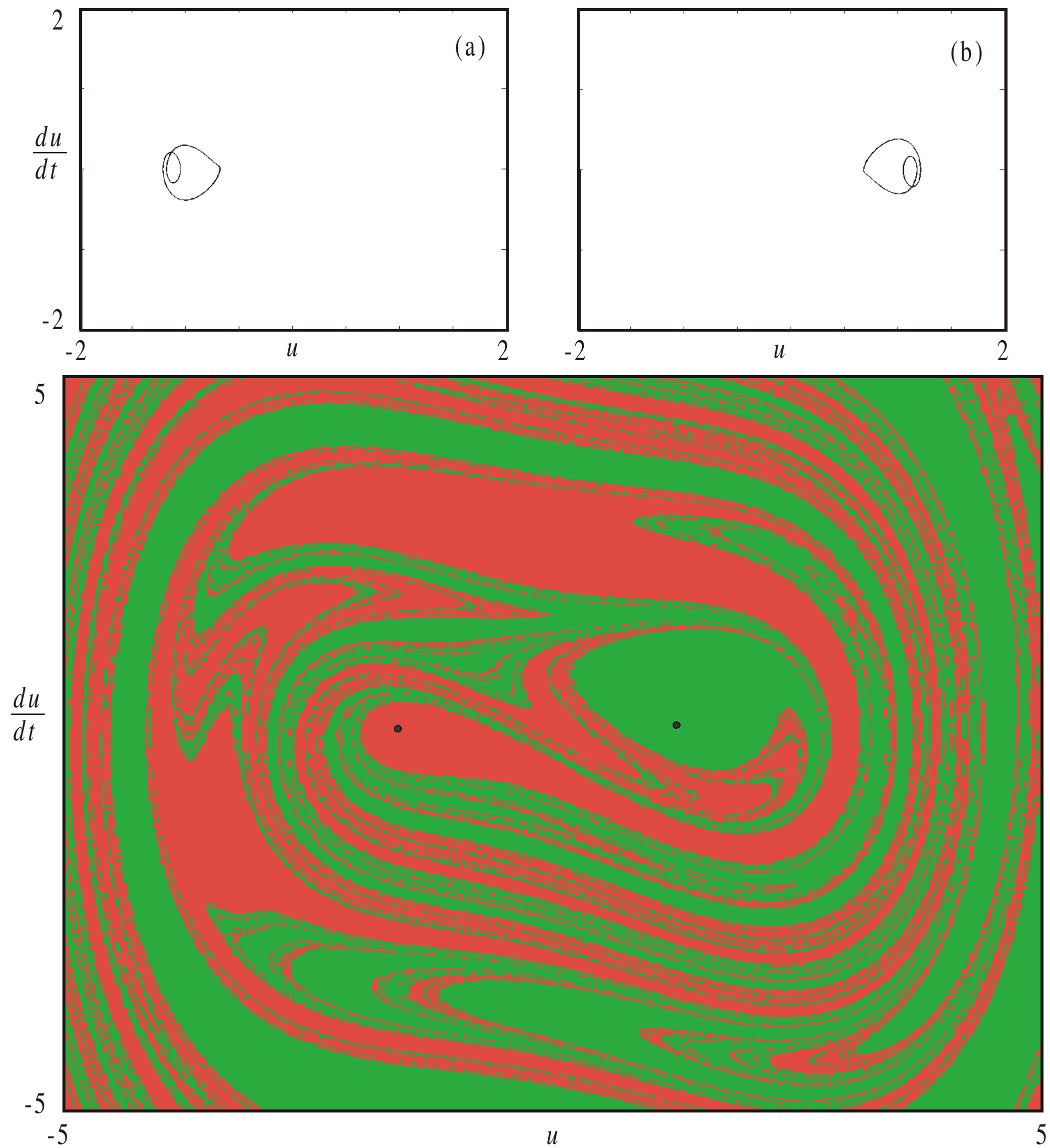


Portrety fazowe: a, b, c

Wykres czasowy: d

Mapy Poincare: e, f

Wykres bifurkacyjny: g


(a)
(b)
2
-2
-2
2
$\frac{du}{dt}$
u
-2
2
5
-5
-5
5