Exercise 13

FREE VIBRATIONS OF THE TWO-DEGREE-OF-FREEDOM SYSTEM

1. Aim of the exercise

Experimental determination of natural frequencies and corresponding modes of the two-pendulum system. Analysis and observation of the beat phenomenon.

2. Theoretical introduction

The system under investigation is shown in Fig. 13.1. Two identical physical pendulums are connected with a spring which is characterized by the stiffness coefficient *k.* At the end of each pendulum of the length l, there is a mass m, which is equal to the mass of the bar m_p. For simplicity, the masses m are treated as the mass-points being at the distance l from the axis of rotation. The spring is mounted at the distance $l_1 = 0.5\,l$.

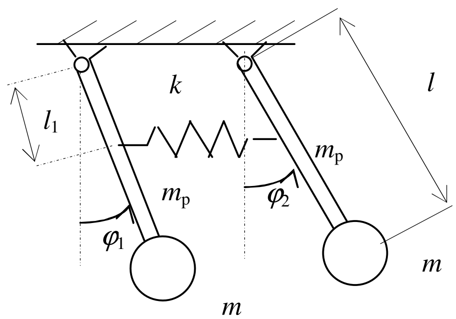

Fig. 13.1. Two-pendulum model

To derive the mathematical model of the two-pendulum system, the second Newton law for rotational motion has been used:

$$B\ddot{\varphi} = \sum M\,, \tag{13.1}$$

where: B – mass moment of inertia of the pendulum,
φ_1, φ_2 – rotation angles of pendulums,
M – moment of the external forces acting on pendulums.

The mass moment of inertia with respect to the rotation axis of each pendulum is expressed by the following relationship:

$$B = \frac{ml^2}{3} + ml^2 = \frac{4}{3}ml^2\,. \tag{13.2}$$

Using (13.1) and (13.2), we can derive the equations of motion of the investigated system in the following form:

$$\frac{4}{3}ml^2\ddot{\varphi}_1 = -\left(mg\frac{l}{2} + mgl\right)\varphi_1 - k\left(\frac{l}{2}\right)^2\varphi_1 + k\left(\frac{l}{2}\right)^2\varphi_2\,.$$

$$\frac{4}{3}ml^2\ddot{\varphi}_2 = -\left(mg\frac{l}{2} + mgl\right)\varphi_2 - k\left(\frac{l}{2}\right)^2\varphi_2 + k\left(\frac{l}{2}\right)^2\varphi_1\,. \tag{13.3}$$

The system of coupled differential equations (13.3) presents the linear form of the mathematical model of the two-pendulum system. After some mathematical transformations, this model has the following form:

$$\frac{4}{3}ml^2\ddot{\varphi}_1 + \left(\frac{3}{2}mgl + \frac{kl^2}{4}\right)\varphi_1 - \frac{kl^2}{4}\varphi_2 = 0;$$
$$\frac{4}{3}ml^2\ddot{\varphi}_2 + \left(\frac{3}{2}mgl + \frac{kl^2}{4}\right)\varphi_2 - \frac{kl^2}{4}\varphi_1 = 0. \qquad (13.4)$$

The solutions to Eqs. (13.4) can be sought as:

$$\varphi_1 = A_1 \sin(\omega_n t + \delta);$$
$$\varphi_2 = A_2 \sin(\omega_n t + \delta). \qquad (13.5)$$

where: ω_n – unknown natural frequency,

A_1, A_2 – amplitudes, which are determined from the initial conditions,

δ – initial phase, which is determined from the initial conditions.

To determine the natural frequencies of the system, we substitute solutions (13.5) into (13.4). Then, the homogeneous system of algebraic equations with respect to the unknown amplitudes A_1 and A_2 is received in the following form:

$$\left(-b_{11}\omega_n^2 + k_{11}\right)A_1 + k_{12}A_2 = 0;$$
$$k_{21}A_1 + \left(-b_{22}\omega_n^2 + k_{22}\right)A_2 = 0, \qquad (13.6)$$

where:

$$b_{11} = b_{22} = \frac{4}{3}ml^2; \quad k_{12} = k_{21} = \frac{kl^2}{4}; \quad k_{11} = k_{22} = \frac{3}{2}mgl + \frac{kl^2}{4}.$$

The non-zero solutions to (13.6) exist when the main determinant is equal to zero. Thus:

$$\begin{vmatrix} -b_{11}\omega_n^2 + k_{11} & k_{12} \\ k_{21} & -b_{22}\omega_n^2 + k_{22} \end{vmatrix} = 0. \qquad (13.7)$$

On the basis of (13.7), we receive the **frequency equation** in the following form:

$$\omega_n^4 - \frac{b_{11}k_{22} + b_{22}k_{11}}{b_{11}b_{22}}\omega_n^2 + \frac{k_{11}k_{22} - k_{12}k_{21}}{b_{11}b_{22}} = 0. \qquad (13.8)$$

The solutions to (13.8) are as follows:

$$\omega_1^2 = \frac{9}{8}\frac{g}{l};$$
$$\omega_2^2 = \frac{9}{8}\frac{g}{l} + \frac{3}{8}\frac{k}{m}. \qquad (13.9)$$

There exist two natural frequencies α_1 and α_2, when the masses m_1 and m_2 are subject to harmonic motion. Thus, there are two **principal modes** of motion, corresponding to the natural frequencies of the system. Substituting ω_1 in (13.6), we obtain:

$$\left(-b_{11}\omega_1^2 + k_{11}\right)A_{11} + k_{12}A_{21} = 0;$$
$$k_{21}A_{11} + \left(-b_{22}\omega_1^2 + k_{22}\right)A_{21} = 0, \qquad (13.10)$$

where: A_{11} - amplitude of the first pendulum, which corresponds to the first principal mode,

A_{21} - amplitude of the second pendulum, which corresponds to the first principal mode.

On the basis of arbitrary equations from system (13.10), the amplitude ratio $\nu_1 = A_{21}/A_{11} = 1$ can be determined. This means that two pendulums move together along the same direction at the same distance. The coupling spring is not stretched or compressed in this process.

In the analogical way for the natural frequency α_2, one receives the amplitude ratio $\nu_2 = A_{22}/A_{12} = -1$. This time both pendulums have the same amplitudes but they are in opposite phases. This motion is symmetrical and the mid-point of the coupling spring does not shift.

The general solution to equations of motion (13.4) is obtained from the superposition of harmonic solutions (13.5). Considering two modes of vibrations, one gets:

$$\begin{aligned}\varphi_1(t) &= A_{11}\sin(\omega_1 t+\delta_1)+A_{12}\sin(\omega_2 t+\delta_2);\\ \varphi_2(t) &= A_{11}\nu_1\sin(\omega_1 t+\delta_1)+A_{12}\nu_2\sin(\omega_2 t+\delta_2).\end{aligned} \tag{13.11}$$

After taking into account the ν_1 and ν_2 values, solutions (13.11) can be written in the following form:

$$\begin{aligned}\varphi_1(t) &= A_{11}\sin(\omega_1 t+\delta_1)+A_{12}\sin(\omega_2 t+\delta_2);\\ \varphi_2(t) &= A_{11}\sin(\omega_1 t+\delta_1)-A_{12}\sin(\omega_2 t+\delta_2),\end{aligned} \tag{13.12}$$

where: A_{11}, A_{12}, δ_1, δ_2 – quantities, which can be determined from the initial conditions.

The question arises: *What initial conditions should be posed to observe the principal modes of vibrations?*

Let us assume the following notations:

$$\begin{aligned}&\varphi_{10}=\varphi_1(t=0); &&\varphi_{20}=\varphi_2(t=0);\\ &\omega_{10}=\left.\frac{d\varphi_1}{dt}\right|_{t=0}; &&\omega_{20}=\left.\frac{d\varphi_2}{dt}\right|_{t=0}.\end{aligned} \tag{13.13}$$

Substituting (13.13) in (13.12), one obtains:

$$\begin{aligned}\varphi_{10} &= A_{11}\sin\delta_1+A_{12}\sin\delta_2;\\ \varphi_{20} &= A_{11}\sin\delta_1-A_{12}\sin\delta_2;\\ \omega_{10} &= A_{11}\alpha_1\cos\delta_1+A_{12}\alpha_2\cos\delta_2;\\ \omega_{20} &= A_{11}\alpha_1\cos\delta_1-A_{12}\alpha_2\cos\delta_2.\end{aligned} \tag{13.14}$$

If we want to observe the first principal mode of vibrations, we should assume $A_{12} = 0$. Then, $\varphi_{10} - \varphi_{20} = 0$, $\varphi_{10} / \varphi_{20} = 1 = \nu_1$ and the initial displacements of pendulums are proportional to the first principal mode. Analogically, $\omega_{10} - \omega_{20} = 0$, and $\omega_{10} = \omega_{20}$ (in particular, one can assume $\omega_{10} = \omega_{20} = 0$).

That means the natural vibrations of the first principal mode can be observed after the introduction of the initial condition $\varphi_{10} = \varphi_{20}$. Then, both pendulums are deflected by the same angle along the same direction. Analogically, one can show that the natural vibrations of the second principal mode can be observed after the introduction of the initial condition $\varphi_{10} = -\varphi_{20}$. The principal modes of vibrations of pendulums are shown in Fig. 13.2.

a) b)

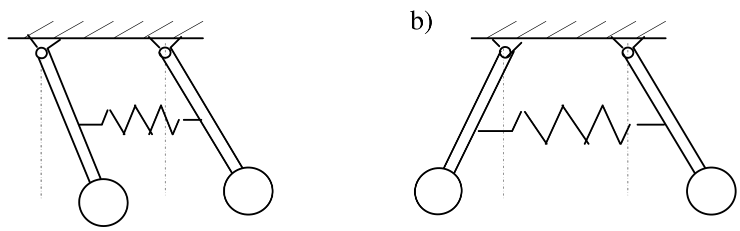

Fig. 13.2. Principal modes of vibrations, a) the first mode, b) the second mode

3. Beat phenomenon

An interesting case can be observed when the initial conditions are: $\varphi_{10} = \varphi_0$ and $\varphi_{20} = 0$, $\omega_{10} = \omega_{20} = 0$. On the basis of Eqs. (13.14), we obtain:

$$
\begin{aligned}
\varphi_0 &= A_{11}\sin\delta_1 + A_{12}\sin\delta_2;\\
0 &= A_{11}\sin\delta_1 - A_{12}\sin\delta_2;\\
0 &= A_{11}\alpha_1\cos\delta_1 + A_{12}\alpha_2\cos\delta_2;\\
0 &= A_{11}\alpha_1\cos\delta_1 - A_{12}\alpha_2\cos\delta_2.
\end{aligned}
\tag{13.15}
$$

From the first two equations (Eqs. 13.15), one receives:

$$\varphi_{10} = 2A_{11}\sin\delta_1\,. \tag{13.16}$$

From the third and fourth equation (Eqs. 13.15), one receives $\delta_1 = \delta_2 = \pi/2$. Taking into account the above dependences, we get:

$$A_{11} = A_{12} = \frac{1}{2}\varphi_0, \tag{13.17}$$

Then, the general solution to equations of motion (13.12) has the following form:

$$
\begin{aligned}
\varphi_1(t) &= \frac{1}{2}\varphi_0\cos\omega_1 t + \frac{1}{2}\varphi_0\cos\omega_2 t;\\
\varphi_2(t) &= \frac{1}{2}\varphi_0\cos\omega_1 t - \frac{1}{2}\varphi_0\cos\omega_2 t.
\end{aligned}
\tag{13.18}
$$

Employing the trigonometric formulas, expressions (13.18) can be presented as follows:

$$
\begin{aligned}
\varphi_1(t) &= \varphi_0\cos\left(\frac{\omega_2-\omega_1}{2}t\right)\cos\left(\frac{\omega_2+\omega_1}{2}t\right);\\
\varphi_2(t) &= \varphi_0\sin\left(\frac{\omega_2-\omega_1}{2}t\right)\sin\left(\frac{\omega_2+\omega_1}{2}t\right).
\end{aligned}
\tag{13.19}
$$

A time history of solutions (13.19) is shown in Fig. 13.3. As can be seen, in the case of two identical conjugated systems having one degree of freedom, the beat phenomenon can be observed in the system under investigation. The energy introduced from the initial conditions is transferred in a periodic way from the first to the second system. This phenomenon is illustrated in Fig. 13.4.

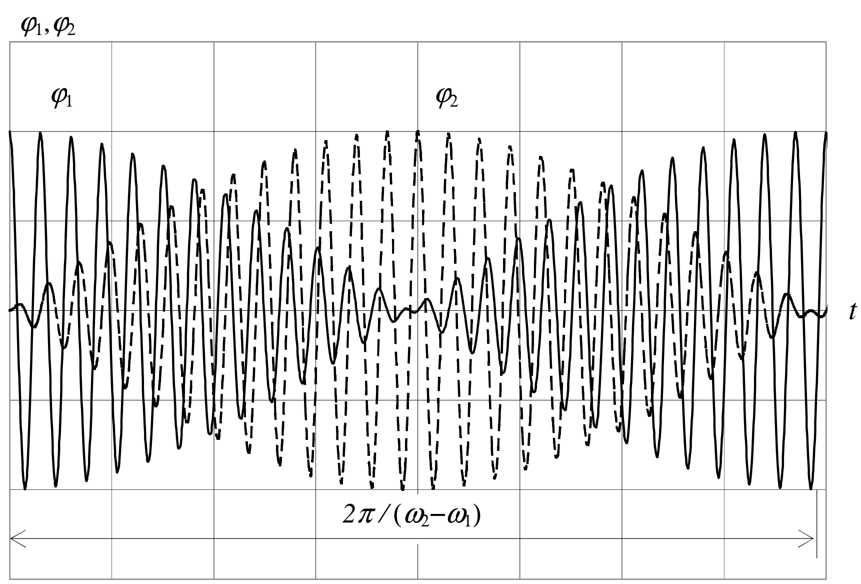

Fig. 13.3. Time history of solutions (13.19)

Initially, the first pendulum moves, whereas the second one is stopped. This motion can be interpreted as the sum of two principal modes with the corresponding natural frequencies α_1 and α_2 (Fig. 13.4a). If the values of natural frequencies are close, certain time is needed (several periods) to observe the phase shift. After this time, the phase shift is equal to 180^0 (Fig. 13.4b). Now, the first pendulum stops, whereas the second one oscillates with the amplitude φ_0. This phenomenon repeats and vibrations transfer from one pendulum to the other one.

4. Measurement device

A scheme of the measurement device is shown in Fig. 13.5. This stand corresponds to the theoretical model which is presented in Fig. 13.1. Both pendulums are suspended on vee blocks and can oscillate only in the vertical plane.

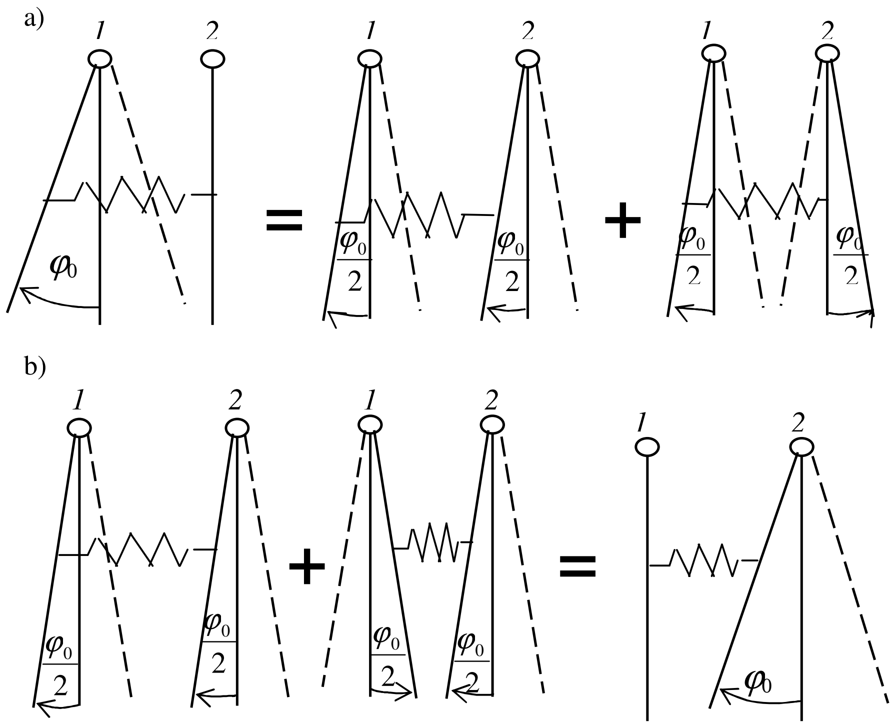

Fig. 13.4. Illustration of the beat phenomenon

5. Course of the exercise

1. Determine experimentally the first natural frequency:
   a) initially both pendulums are deflected by the same angle along the same direction,
   b) measure 20 periods of natural vibrations using a stop-watch,
   c) measurement of 20 periods should be repeated twice,
   d) calculate the average value of the period,
   e) calculate the first natural frequency on the basis of the period of natural vibrations.

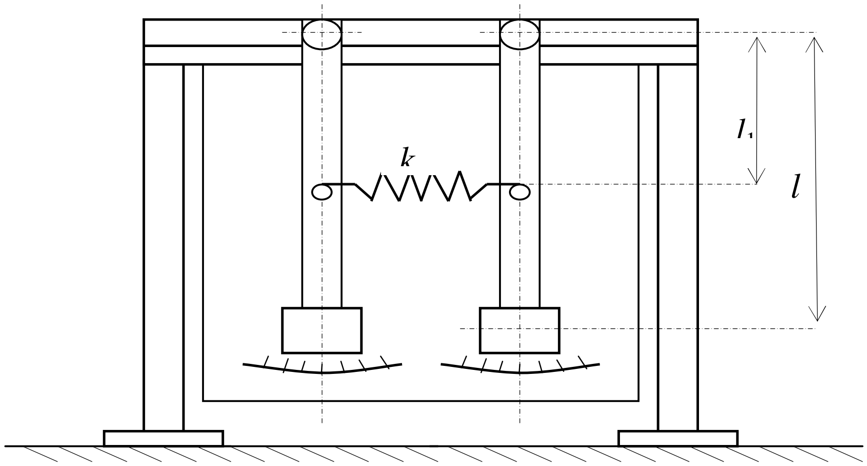

Fig. 13.5. Scheme of the measurement device

2. Determine experimentally the second natural frequency:
   a) initially both pendulums are deflected by the same angle but along opposite directions,
   b) measure 20 periods of natural vibrations using a stop-watch,
   c) measurement of 20 periods should be repeated twice,
   d) calculate the average value of the period,
   e) calculate the second natural frequency on the basis of the period of natural vibrations.

3. Determine experimentally the beat frequency:
   a) initially one pendulum is deflected by a small angle ($\varphi_0 \le 5^0$),
   b) measure 5 periods of beating with a stop-watch,
   c) repeat twice the measurement of 5 periods,
   d) calculate the average value of the beat period,
   e) calculate the beat frequency on the basis of the beat period.

4. Compare the experimental and theoretical results:
   a) calculate the theoretical natural frequencies α_1 and α_2 on the basis of Eqs. (13.9),
   b) calculate the theoretical beat frequency $\alpha_b = \alpha_2 - \alpha_1$,
   c) analyse the experimental and theoretical results.

6. Laboratory report should contain:
1. Aim of the exercise.
2. Experimental and calculation results presented in the following table:

Determination of the first natural frequency
Time of 20 periods of natural vibrations: T_{11} = ... [s]; T_{12} = ... [s]; T_{13} = ... [s]
The average value of the period: $T_1 = \dfrac{T_{11}+T_{12}+T_{13}}{3\times 20} =$
The first natural frequency value: $\alpha_1 = \dfrac{2\pi}{T_1} =$
Determination of the second natural frequency
Time of 20 periods of natural vibrations:

T_{21} = ... [s]; T_{22} = ... [s]; T_{23} = ... [s] The average value of the period: $T_2 = \dfrac{T_{21}+T_{22}+T_{23}}{3\times 20} =$ The second natural frequency value: $\alpha_2 = \dfrac{2\pi}{T_2} =$
Determination of the beat frequency
Time of 5 periods of beating: T_{b1} = ... [s]; T_{b2} = ... [s]; T_{b3} = ... [s] The average value of the period: $T_b = \dfrac{T_{b1}+T_{b2}+T_{b3}}{3\times 5} =$ The beat frequency value: $\alpha_b = \dfrac{2\pi}{T_b} =$

3. Results of the theoretical calculations:
   Numerical data:
   - stiffness of the spring: k = ……………….… ;
   - mass at the end of the bar: m =…………….. ;
   - length of the bar: l = ……………………..……

   The first natural frequency calculation:
   $\alpha_1 = \sqrt{\frac{9}{8}\frac{g}{l}} =$
   The second natural frequency calculation:
   $\alpha_2 = \sqrt{\frac{9}{8}\frac{g}{l} + \frac{3}{8}\frac{k}{m}} =$
   The beat frequency calculation:
   $\alpha_d = \alpha_2 - \alpha_1 =$

4. Comparison of the experimental and theoretical results.
5. Conclusions and remarks.

References

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