## Exercise 6

## TRANSFORMATION OF STATE VARIABLES

### 1. Aim of the exercise

Determination of the state-space description of a two-degree-of-freedom system in various sets of state variables with the MATLAB software package.

### 2. Theoretical introduction

Sometimes state variables used in the original formulation of the system dynamics are not so convenient as another set of state variables. Instead of reformulating the mathematical model of the system, it is possible to transform the state matrix **A**, the input matrix **B**, the output matrix **C** and the transition matrix **D** of the original formulation to a new set of matrices. The change of variables is represented by the linear transformation:

$$\mathbf{z}(t) = \mathbf{T}\,\mathbf{x}(t) \tag{6.1}$$

where: $\mathbf{z}(t)$ – state vector in the new formulation,
$\mathbf{x}(t)$ – state vector in the original formulation,
$\mathbf{T}$ – nonsingular transformation matrix ($k \times k$).

On the basis of (6.1), one can write:

$$\mathbf{x}(t) = \mathbf{T}^{-1}\,\mathbf{z}(t) \tag{6.2}$$

Let us assume that **T** is a constant matrix. The original system is defined by the state equation and the output equation in the following form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \tag{6.3}$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) \tag{6.4}$$

where: $\mathbf{x}(t)$ - $n$-dimensional state vector,
$\mathbf{u}(t)$ - $m$-dimensional input signal vector,
$\mathbf{y}(t)$ - $r$-dimensional output signal vector,
**A** - state matrix ($n \times n$),
**B** - input matrix ($n \times m$),
**C** - output matrix ($r \times n$),
**D** - transition matrix ($r \times m$).

By substitution of (6.2) into (6.3) and (6.4), we obtain:

$$\dot{\mathbf{z}}(t) = \mathbf{TAT}^{-1}\,\mathbf{z}(t) + \mathbf{TB}\,\mathbf{u}(t) \tag{6.5}$$

$$\mathbf{y}(t) = \mathbf{CT}^{-1}\,\mathbf{z}(t) + \mathbf{D}\,\mathbf{u}(t) \tag{6.6}$$

Using Eqs. (6.3) and (6.4), one can write the normal form of the state-space description of the new formulated system:

$$\dot{\mathbf{z}}(t) = \overline{\mathbf{A}}\,\mathbf{z}(t) + \overline{\mathbf{B}}\,\mathbf{u}(t) \tag{6.7}$$

$$\mathbf{y}(t) = \overline{\mathbf{C}}\,\mathbf{z}(t) + \overline{\mathbf{D}}\,\mathbf{u}(t) \tag{6.8}$$

where: $\overline{\mathbf{A}} = \mathbf{TAT}^{-1}$; $\overline{\mathbf{B}} = \mathbf{TB}$; $\overline{\mathbf{C}} = \mathbf{CT}^{-1}$; $\overline{\mathbf{D}} = \mathbf{D}$. (6.9)

The state matrix of the transformed system $\overline{\mathbf{A}} = \mathbf{TAT}^{-1}$ is said to be similar to the state matrix **A** of the original system. Similar matrices have the same characteristic polynomial.

### 3. Example of the state variable transformation

Figure 6.1 shows a dynamic system consisting of two discs which are coupled with a spring of the stiffness coefficient $k$. On the basis of the Newton's law, one can derive the differential equations of motion in the following form:

$$B_{01}\ddot{\varphi}_1 = -kr_1^2\varphi_1 + kr_1r_2\varphi_2 + u_1, \qquad \text{(a)}$$

$$B_{02}\ddot{\varphi}_2 = -kr_2^2\varphi_2 + kr_1r_2\varphi_1 + u_2. \qquad \text{(b)}$$

where: $r_1$, $r_2$ – radii of the first and second disc, respectively,
$B_{01}$, $B_{02}$ – mass moments of inertia of discs,
$\varphi_1$, $\varphi_2$ – angular displacements (output signals),
$u_1$, $u_2$ – externally applied moments (input signals).

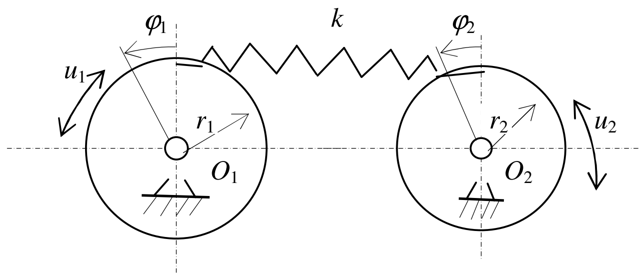


Fig. 6.1. Two-disc dynamic system

Defining the state as:

$$\mathbf{x} = [\varphi_1,\ \dot{\varphi}_1,\ \varphi_2,\ \dot{\varphi}_2]^T \qquad \text{(c)}$$

one receives the following matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{kr_1^2}{B_{01}} & 0 & \dfrac{kr_1r_2}{B_{01}} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{kr_1r_2}{B_{02}} & 0 & -\dfrac{kr_2^2}{B_{02}} & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ \dfrac{1}{B_{01}} & 0 \\ 0 & 0 \\ 0 & \dfrac{1}{B_{02}} \end{bmatrix} \qquad \text{(d)}$$

The characteristic polynomial of the state matrix $\mathbf{A}$ takes the following form:

$$|s\mathbf{I} - \mathbf{A}| = s^4 + k\left(\frac{r_1^2}{B_{01}} + \frac{r_2^2}{B_{02}}\right)s^2 \qquad \text{(e)}$$

Now, let us suppose that it might be more convenient to define the motion of the system by the motion of the external point of the first disc:

$$y = \varphi_1 r_1, \qquad \text{(f)}$$

and the difference between the positions of the discs:

$$\delta = \varphi_1 - \varphi_2. \qquad \text{(g)}$$

The new state vector is:

$$\mathbf{z} = [y,\ \dot{y},\ \delta,\ \dot{\delta}]^T \qquad \text{(h)}$$

On the basis of (f) and (g), one can define the transformation matrix and its inversion:

$$\mathbf{T} = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}; \quad \mathbf{T}^{-1} = \begin{bmatrix} 1/r_1 & 0 & 0 & 0 \\ 0 & 1/r_1 & 0 & 0 \\ 1/r_1 & 0 & -1 & 0 \\ 0 & 1/r_1 & 0 & -1 \end{bmatrix} \qquad \text{(i)}$$

The state matrix of the transformed system:

$$\overline{\mathbf{A}} = \mathbf{TAT}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ k\left(\dfrac{r_1 r_3 - r_1^2}{B_{01}}\right) & 0 & -\dfrac{k r_1^2 r_2}{B_{01}} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k}{B_{01}}(r_2 - r_1) + \dfrac{k r_2}{B_{02}}\left(\dfrac{r_2}{r_1} - 1\right) & 0 & -k r_2\left(\dfrac{r_1}{B_{01}} + \dfrac{r_2}{B_{02}}\right) & 0 \end{bmatrix} \qquad \text{(j)}$$

It is easy to check that the characteristic polynomial of the matrix (j) has the form (e) – the same as the characteristic polynomial of the state matrix **A**. The input matrix and the output matrix of the transformed system have the following forms:

$$\overline{\mathbf{B}} = \mathbf{TB} = \begin{bmatrix} 0 & 0 \\ \dfrac{r_1}{B_{01}} & 0 \\ 0 & 0 \\ \dfrac{1}{B_{01}} & -\dfrac{1}{B_{02}} \end{bmatrix}; \quad \overline{\mathbf{C}} = \mathbf{CT}^{-1} = \begin{bmatrix} \dfrac{1}{r_1} & 0 & 0 & 0 \\ \dfrac{1}{r_1} & 0 & -1 & 0 \end{bmatrix} \qquad \text{(k)}$$

## 4. Course of the exercise

Figure 6.2 shows a two-degree-of-freedom system which is considered in the exercise. Two masses are connected with a spring. The masses can slide horizontally without friction. Notations: $u_1$, $u_2$ – externally applied forces (input signals), $y_1$, $y_2$ – displacements (output signals).

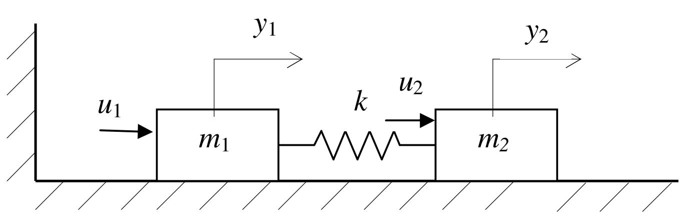


Fig. 6.2. Two-degree-of-freedom system

During the exercise, one ought to carry out the following operations:

a) derive the differential equations of motion,
b) define the state of the system,
c) determine the state equation and the output equation using the MATLAB software package,
d) determine the characteristic polynomial of the state matrix,
e) define new state variables and the transformation matrix,
f) determine the state equation and the output equation of the transformed system with MATLAB,
g) verify the results using MATLAB procedures.

## 5. Laboratory report should contain:

1) Aim of the exercise.
2) Mathematical model of the original system in the form of second-order differential equations and in the form of state-space descriptions.
3) Mathematical model of the transformed system in the form of state-space descriptions.
4) Results of the computer calculations.
5) Conclusions and remarks.

## References


1. Ogata K.: *Modern Control Engineering,* IV-th Edition, Prentice Hall, 2002.
2. Friendland B.: *Control System Design,* McGraw-Hill Book Company, 1987.
3. Wolovich W.A.: *Automatic Control Systems, Basic Analysis and Design,* Harcour Brace College Publishers, 1994.