Excercise 2a

Determination of the minimum period of free oscillations of a physical pendulum

Aim of the exercise

The aim of the exercise is to determine the position of the axis of rotation of the physical pendulum for which the period of free vibrations of this pendulum is the shortest.

Rig description

The subject of the considerations is a physical pendulum in the form of a homogeneous rod with a length of *l*=1.03 meters, capable of swinging around the axis of rotation A. The pendulum is shown in Figure 1.

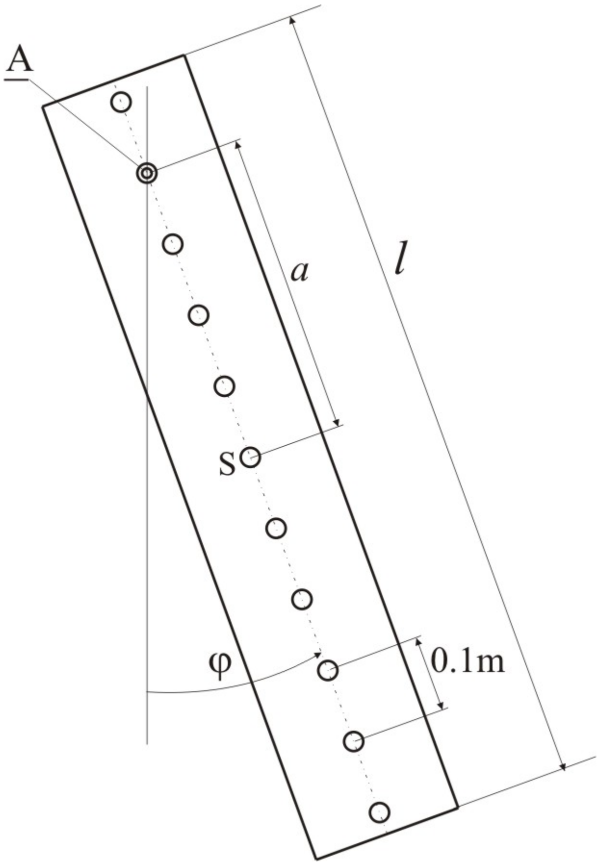


Figure 1

Based on the dynamic equation of the rotational motion of a body (equation of the equilibrium of the moments of forces with respect to the axis of rotation, taking into account the moment of inertia simply resulting from the Newton's Second Law of Dynamics), the following equation of the pendulum motion can be formulated:

$$B_A\ddot{\phi} + mga\phi = 0 \qquad (1)$$

where:
B_A – mass moment of inertia with respect to the axis of rotation,
m - mass of the pendulum,
g – gravity acceleration = 9.81 m/s^2,
a – distance between actual axis of rotation and the one related to the center of gravity (CG),
φ - angle of rotation (varying).

Period of free vibration of the pendulum is described by the formula being outcome of the solution of Eq. (1):

$$T_A = 2\pi\sqrt{\frac{B_A}{mga}}. \qquad (2)$$

Here you have to answer the fundamental question – how to solve the above equation and arrive to formula (2)…

As an arbitrary axis of rotation for the discussed pendulum can be located in any position at its length, then you must recall to the formula connecting dependence of mass moment of inertia to any position of the rotation axis also known as Steiner theorem on parallel axes in the case.

$$B_A = B_S + ma^2 \tag{3}$$

where:

B_S – mass moment of inertia at rotation axis its center of gravity.

The B_S moment is unique for a pendulum and remain constant for the model which represents a uniform beam and is represented by

$$B_S = \frac{ml^2}{12} \tag{4}$$

Applying eq. (3) to Eq. (2) we arrive to the final form of formula describing value of period *T* depending on its length and position of the support:

$$T_A(a) = 2\pi\sqrt{\frac{B_S + ma^2}{mga}} = 2\pi\sqrt{\frac{\frac{l^2}{12} + a^2}{ga}} \tag{5}$$

The formula above may be interpreted as a function *T(a)* of a varying argument *a* (independent variable) – the distance between point of support and position of the beam center of gravity. The formula above directly shows the smaller value of *a* (axis of rotation tends closer to the position of CG) the larger value of the period *T* of the pendulum's oscillations (tends to infinity). The numerator of the fraction under the root square tends to a constant while a approaches zero) while the denominator simply reaches zero…
Similarly, it is easy to conclude that when the distance a increases to infinity (the axis of rotation moves away to the center of mass by an unlimited distance many times greater than *l*), the period of vibration also increases – the value of a becomes dominant over *l* and since the numerator is of higher order than the denominator. The question arises for what value of the argument a function (5) has an extreme value and how much is that.
To find the answer, one has to recall the basic knowledge of function studying which states the maxima and minima (the respective plurals of maximum and minimum) of a function, known collectively as extrema (the plural of extremum), are the largest and smallest value of the function, either within a given range (the local or relative extrema), or on the entire domain (the global or absolute extrema). Finding global maxima and minima is the goal of mathematical optimization. If a function is continuous on a closed interval, then by the extreme value theorem, global maxima and minima exist. Furthermore, a global maximum (or minimum) either must be a local maximum (or minimum) in the interior of the domain, or must lie on the boundary of the domain. So a method of finding a global maximum (or minimum) is to look at all the local maxima (or minima) in the interior, and also look at the maxima (or minima) of the points on the boundary, and take the largest (or smallest) one. This is known as Fermat's theorem which is central to the calculus method of determining maxima and minima: in one dimension, one can find extrema by simply computing the stationary points (by computing the zeros of the derivative), the non-differentiable points, and the boundary points, and then investigating this set to determine the extrema.

One can distinguish whether a critical point is a local maximum or local minimum by using the first derivative test, second derivative test, or higher-order derivative test, given sufficient differentiability.

To determine the answer to the above stated question we need to differentiate (5) over the variable a, which gives

$$\frac{\mathrm{d}T_A(a)}{\mathrm{d}a} = \frac{2\pi}{2\sqrt{\frac{B_S + ma^2}{mga}}}\frac{2m^2a^2g - (B_S + ma^2)mg}{m^2g^2a^2} = \frac{\pi\left(a^2 - \frac{l^2}{12}\right)}{\sqrt{\frac{gl^2a^3}{12} + ga^5}} \qquad (6)$$

It is rather easy to spot the above new function reaches zero for

$$\frac{\mathrm{d}T_A(a)}{\mathrm{d}a} = 0 \Rightarrow a = \frac{l}{\sqrt{12}} \qquad (7)$$

And this is the only extreme value of the function – your logic suggests this is a minimum. So, the period of oscillations T_A is shortest for the distance a equal to the value of the pendulum length divided by square root of 12. Substituting l with this value we can calculate

$$T_{A\min} = 2\pi\sqrt{\frac{\frac{ml^2}{12} + m\frac{l^2}{12}}{mg\frac{l}{\sqrt{12}}}} = 2\pi\sqrt{\frac{\sqrt{12}l}{6g}} \qquad (8)$$

Yet we need to proof the extreme found is minimum and not a maximum. To do that we differentiate the first derivative of *T(a)* over *a* again:

$$\frac{\mathrm{d}^2T_A(a)}{\mathrm{d}a^2} = \pi\frac{2a\sqrt{\frac{gl^2a^3}{12} + ga^5} - \left(a^2 - \frac{l^2}{12}\right)\frac{5ga^4 + \frac{3gl^2a^2}{12}}{2\sqrt{\frac{gl^2a^3}{12} + ga^5}}}{\frac{gl^2a^3}{12} + ga^5} \qquad (9)$$

The obtained formula looks a bit complicated, BUT it is rather easy to find for $a = \frac{l}{\sqrt{12}}$ the part in the brackets in the numerator is equal zero, so all the second term there is zero. The first part must be positive as all its symbols represent physical (and positive) values, so

$a = \frac{l}{\sqrt{12}} \Rightarrow \frac{\mathrm{d}^2T_A(a)}{\mathrm{d}a^2} > 0$ the second derivative at this point possess positive value, so the period *T(a)* is minimal!

How to perform the practical test:

Using a time measuring device determine how long it takes to realize 20 periods of the pendulum's free oscillations supported at least in five different points located by 0.50, 0.40, 0.30, 0.20 and 0.10 meter form the CG. After dividing the obtained time values by 20 you get respective period values T_A as a function of a.
Using formula (5) calculate theoretical value of T_A at different a values given in the second table below.
Draw theoretical graph T_A (a). Add experimentally obtained values with separate symbols.
Find all your conclusions from the performed actions.
Realize – in many cases you need to determine some properties of elements which later can be used in calculations/simulations – prediction of behavior of the dynamical behavior of systems.

First and Family names: ……………..…….. Group:………… Mark:………..

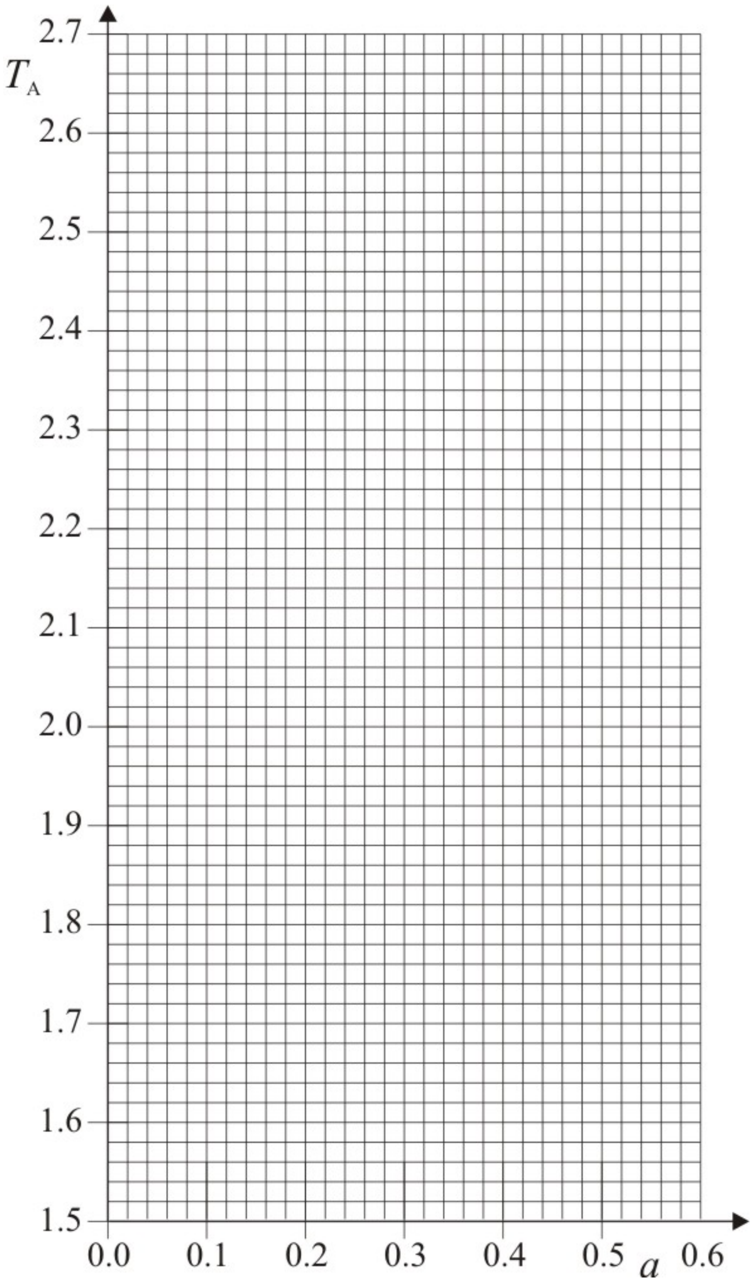
Report from Exercise 2A
Determination of the minimum period of free oscillations of a physical pendulum

Experimenta results:

a	$20T_A$	T_A
0.10		
0.20		
0.30		
0.40		
0.50		

Theoretical calculations

a	T_A
0.05	
0.10	
0.15	
0.20	
0.25	
0.30	
0.35	
0.40	
0.45	
0.50	
0.55	
0.60	



Formula applied:

$$T_A(a) = 2\pi\sqrt{\frac{\frac{l^2}{12}+a^2}{ga}}, \quad l = 1.03\text{m}, \quad g = 9.81\frac{\text{m}}{\text{s}^2}$$